The Apollonian staircase

James Rickards

CU Boulder

james.rickards@colorado.edu

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Apollonius of Perga

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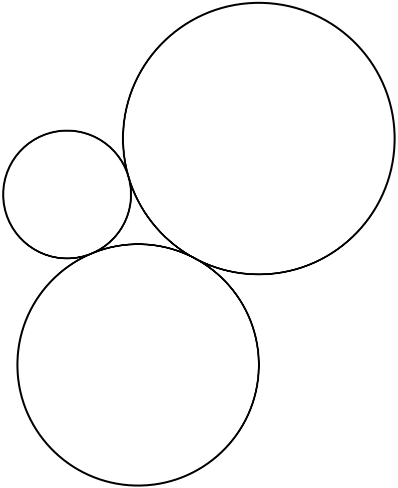
Originally studied by Apollonius in the lost book "De tactionibus".

Example 1



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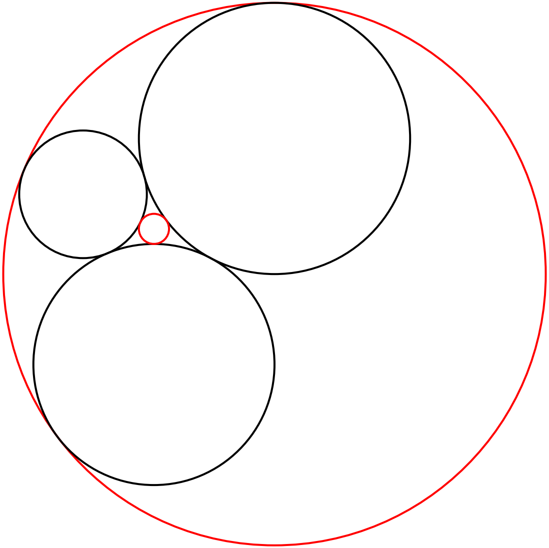
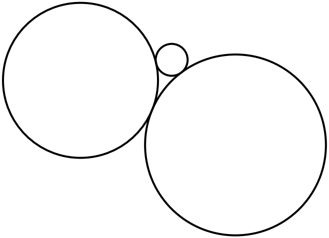
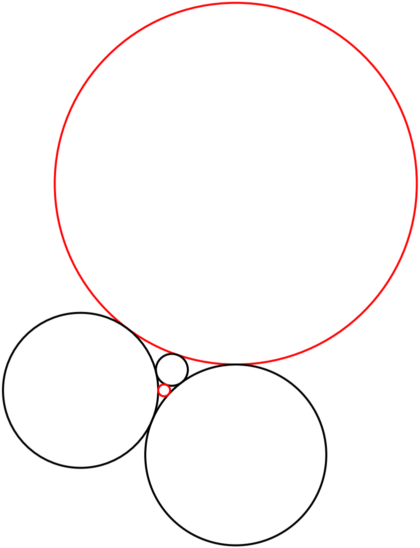


Example 2



Example 2



Apollonian circle packing

- Start with three mutually tangent circles;

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- Start with three mutually tangent circles;
- Draw the two circles tangent to all three;

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- Start with three mutually tangent circles;
- Draw the two circles tangent to all three;
- This creates six more triples of mutually tangent circles. Repeat!

Apollonian circle packing - example

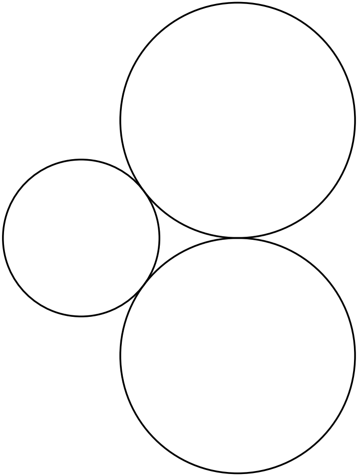
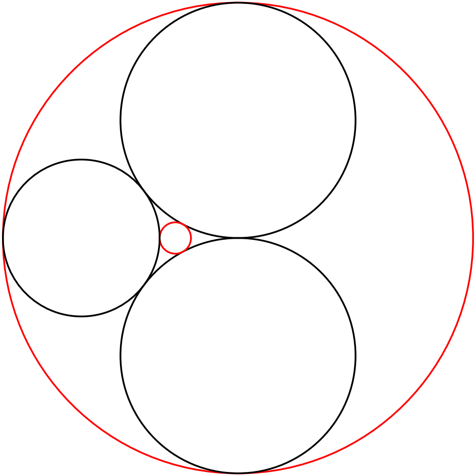


Figure 1: Generation 0

Apollonian circle packing - example



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Apollonian circle packing - example

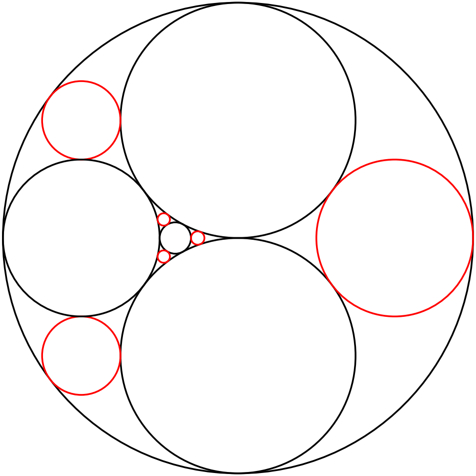

Figure 1: Generation 2

Apollonian circle packing - example

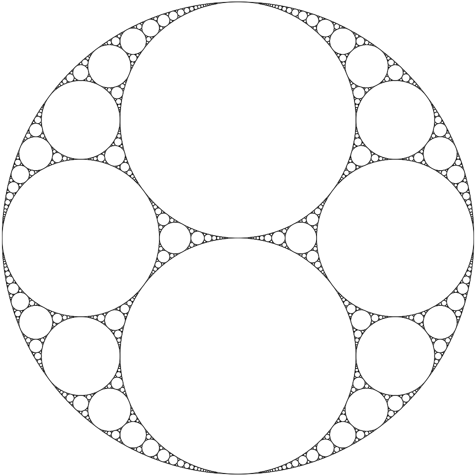

Figure 1: Radius $\geq \frac{1}{500}$.

René Descartes

Definition

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Theorem (Descartes, 1643)

Let four mutually tangent circles have curvatures a, b, c, d. Then

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Definition

We call (a, b, c, d) a Descartes quadruple.

Types of packings

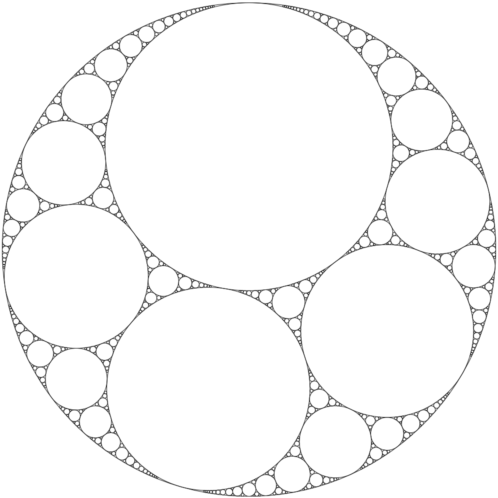

Figure 2: Bounded: $(-7, 12, 17, 20)$

Types of packings

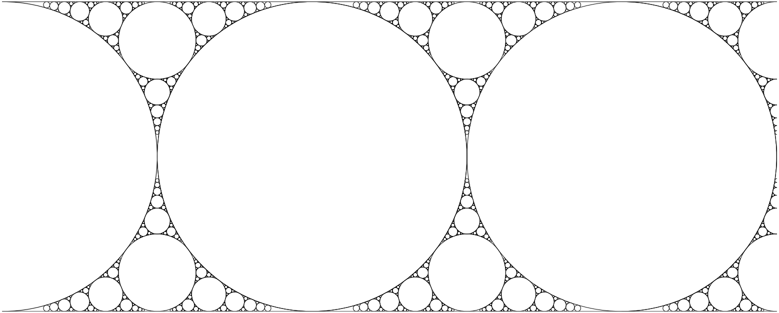

Figure 2: Strip: $(0, 0, 1, 1)$

Types of packings

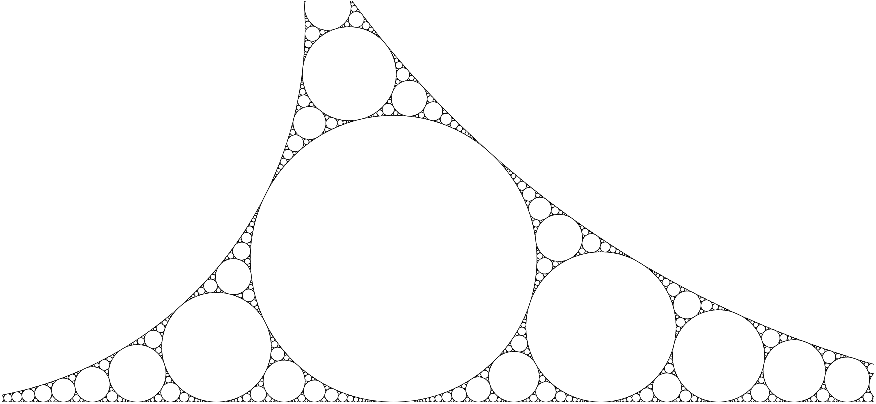

Figure 2: Half-plane: $(0, 1, \phi + 1, 3\phi + 2)$

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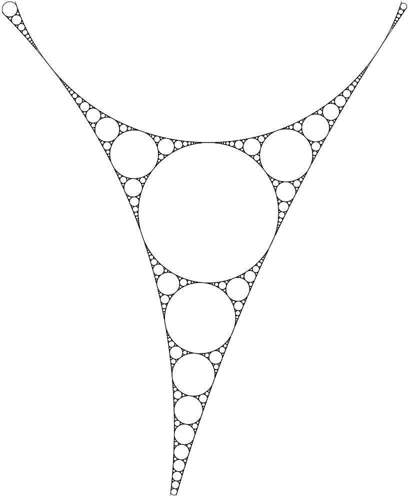

Figure 2: Full-plane: $(1, \phi - \sqrt{\phi}, (\phi - \sqrt{\phi})^2, (\phi - \sqrt{\phi})^3)$

Integral theory

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- Therefore if $(a, b, c, d) \in \mathbb{Z}^4$, then $d' \in \mathbb{Z}$. In particular, all curvatures in the packing are integers.
- We normally restrict to primitive packings, i.e. $\gcd(a, b, c, d) = 1$.

Example

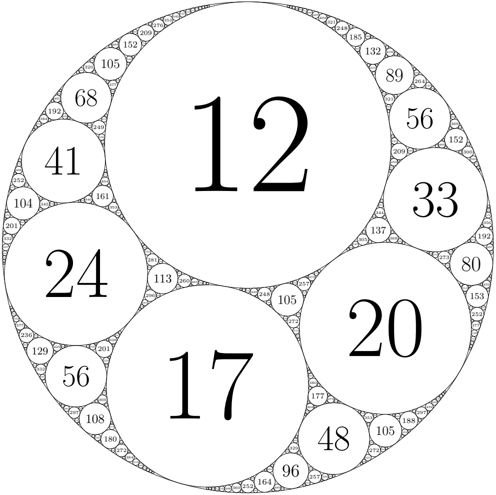

Figure 3: $(-7, 12, 17, 20)$, circles labelled by curvature

Apollonian group

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- Words in S_1, S_2, S_3, S_4 allow you to move between the different Descartes quadruples in the same Apollonian circle packing.

Example

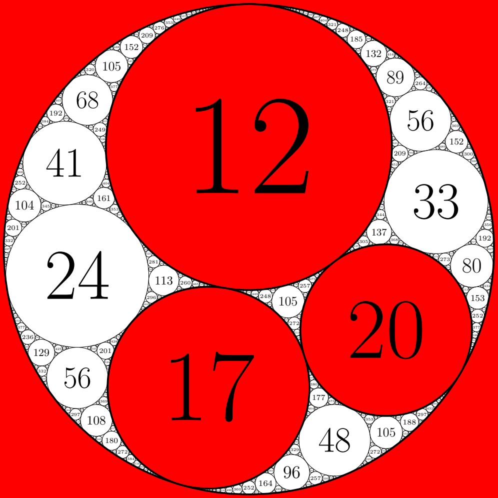

Figure 4: $(-7, 12, 17, 20)$

Example

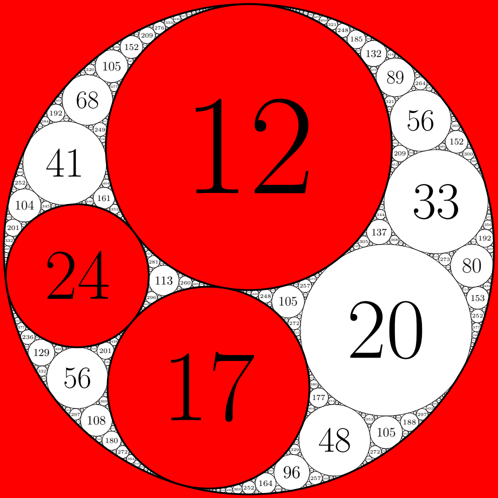

Figure 5: $S_4(-7, 12, 17, 20) = (-7, 12, 17, 24)$

Example

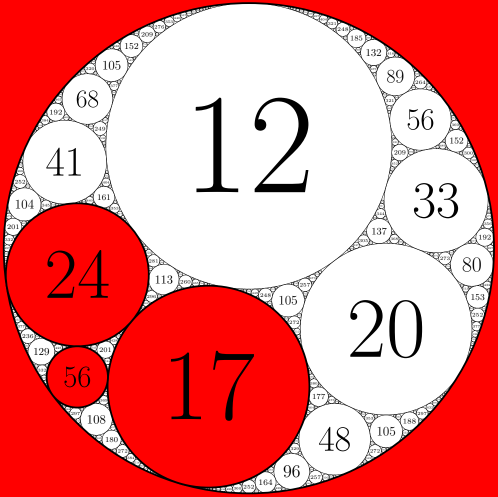

Figure 6: $S_2S_4(-7,12,17,20) = (-7,56,17,24)$

Example

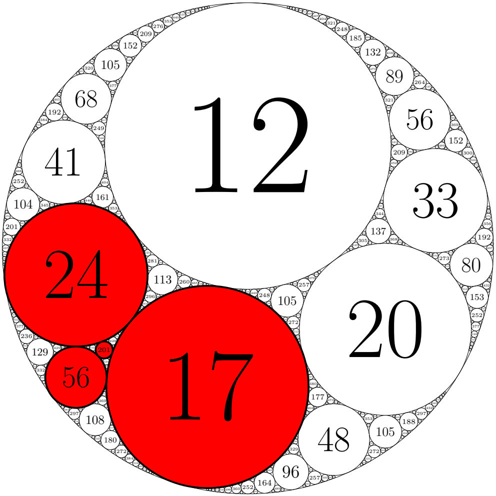

Figure 7: $S_1 S_2 S_4(-7, 12, 17, 20) = (201, 56, 17, 24)$

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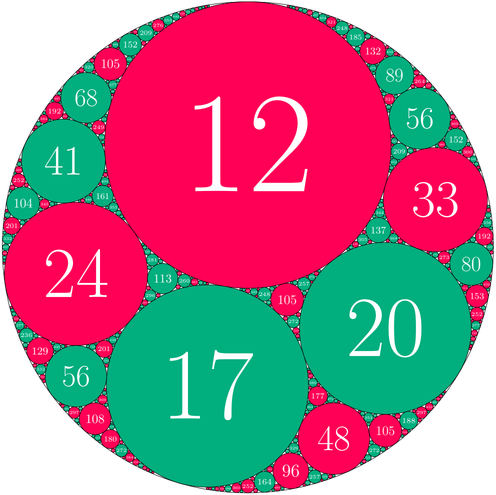

Figure 8: $(-7, 12, 17, 20)$ coloured modulo 3

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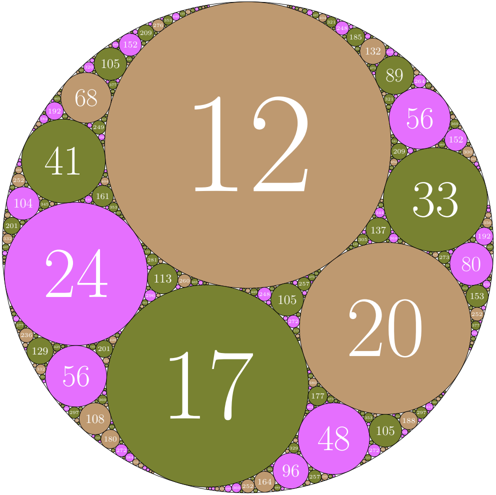

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Every sufficiently large curvature that is admissible modulo 24 occurs in the packing.

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The best known result is density 1; see Bourgain-Kontorovich ([BK14]) or Fuchs-Stange-Zhang ([FSZ19]).

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Definition (From [GLM+03])

Let $[A, B, C] = AX^2 + BXY + CY^2$ be a non-zero binary quadratic form of discriminant $-4n^2$ for some $n \in \mathbb{R}$, where $A, C \geq 0$. Define

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Then $\theta([A, B, C])$ is a Descartes quadruple containing a circle of curvature n. The inverse to θ is

$$\phi(n, a, b, c) = [n + a, n + a + b - c, n + b].$$

Parameter space

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Let $\mathbf{q} = (n, a, b, c)$ be a Descartes quadruple, and define $p_{\mathbf{q}} \in \mathbb{P}^1(\mathbb{C})$ to be a root of the quadratic form $\phi(\mathbf{q})$. If $n > 0$ we take the upper half plane root, and if $n < 0$ we take the lower half plane root.

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Proposition

For a fixed $n \in \mathbb{R}^+$, Descartes quadruples with n as the first curvature biject with $\mathbb{H}$ under the correspondence $\mathbf{q} \to p_{\mathbf{q}}$.

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Applying a word in S_2, S_3, S_4 or permuting the last 3 curvatures changes **q**, but retains the same circle of curvature n in the first position.

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Proposition

Taking a circle of curvature n inside an Apollonian circle packing is equivalent to taking a $\mathrm{PGL}(2,\mathbb{Z})$ *equivalence class of quadratic forms of discriminant* $-4n^2$.

Integral theory

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In particular,

$$|\,\mathrm{ID}(n)| = h^{\pm}(-4n^2) = \frac{1}{2}\left(h^{+}(-4n^2) + a(-4n^2)\right),$$

where $h^{+}()$ is the narrow class number, and $a()$ counts ambiguous classes.

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This is a finite number that tends to ∞ as $n \to \pm\infty$.

All primitive integral packings containing 11

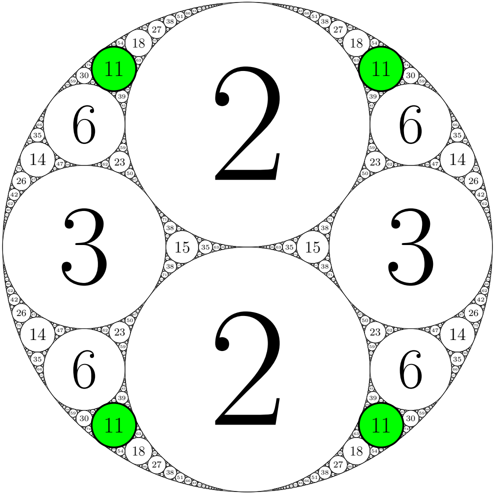

Figure 8: $(-1, 2, 2, 3)$

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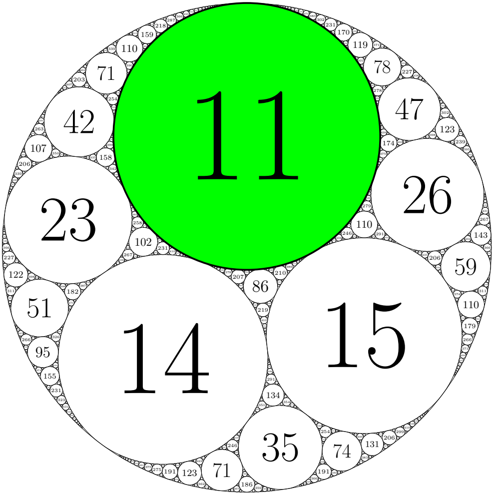

Figure 8: $(-6, 11, 14, 15)$

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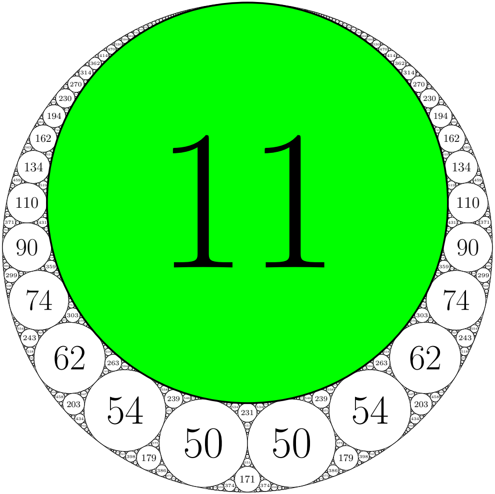

Figure 8: $(-9, 11, 50, 50)$

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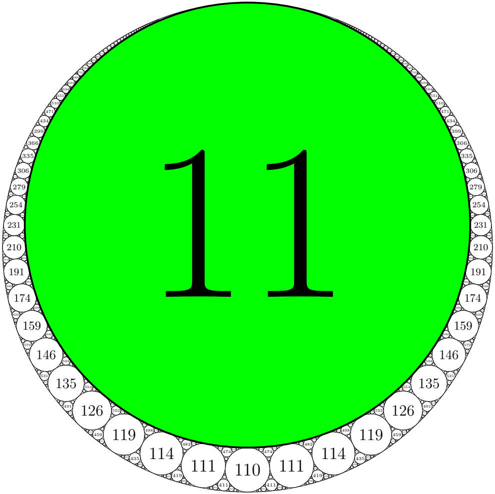

Figure 8: $(-10, 11, 110, 111)$

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Let $\mathrm{MC}(\mathbf{q})$ be the negative of the minimal curvature in the packing.

If the first curvature in $\mathbf{q}$ is $n \in \mathbb{R}^+$, define the height of $\mathbf{q}$ to be

$$H(\mathbf{q}) = \frac{\mathrm{MC}(\mathbf{q})}{n} \in [0, 1).$$

Relative minimal curvature

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- $\mathrm{RMC}(11) = \left\{\frac{1}{11}, \frac{6}{11}, \frac{9}{11}, \frac{10}{11}\right\}$.
- What is the distribution of $\mathrm{RMC}(n)$ as $n \to \infty$?

Staircase

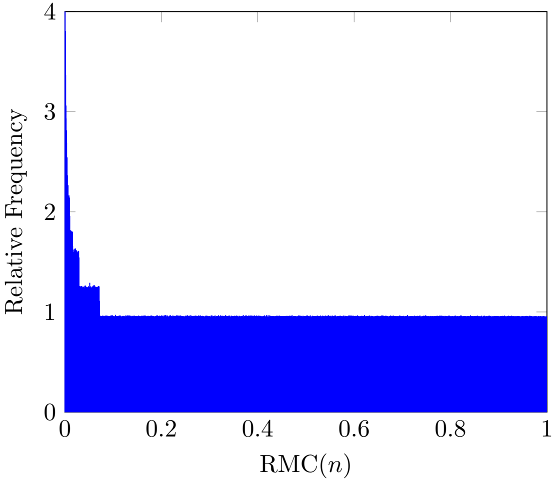

Figure 9: Histogram for $n = 33920039$; 8480011 data points in 2000 bins.

Spiky staircase

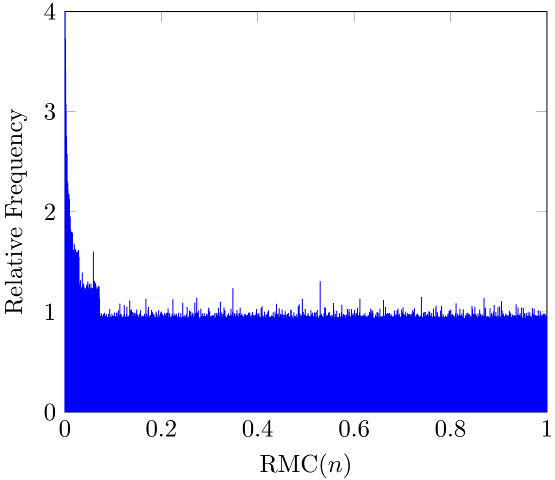

Figure 10: Histogram for $n = 42728555$; 8480008 data points in 2000 bins.

Main results

Theorem (R., 2022)

As $n \to \infty$, the distribution $\mathrm{RMC}(n)$ *tends to the Apollonian staircase.*

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Table 1: The first 5 stairs

Cutoff	Height
1	0.9549296586
0.0717967697	0.2886751346
0.0294372515	0.3535533906
0.0161332303	0.1936491673
0.0102051443	0.3674234614

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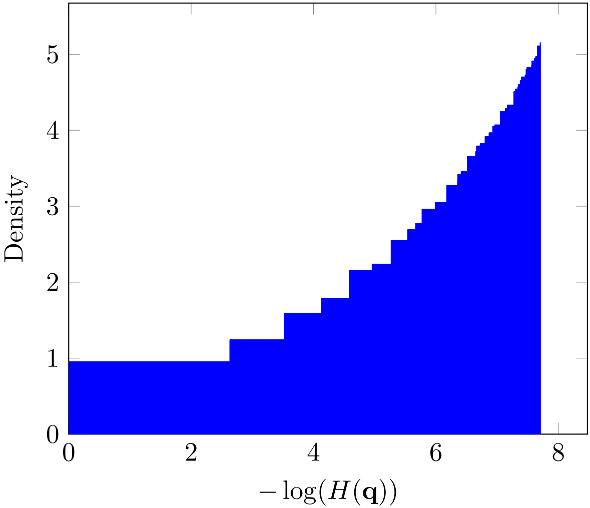
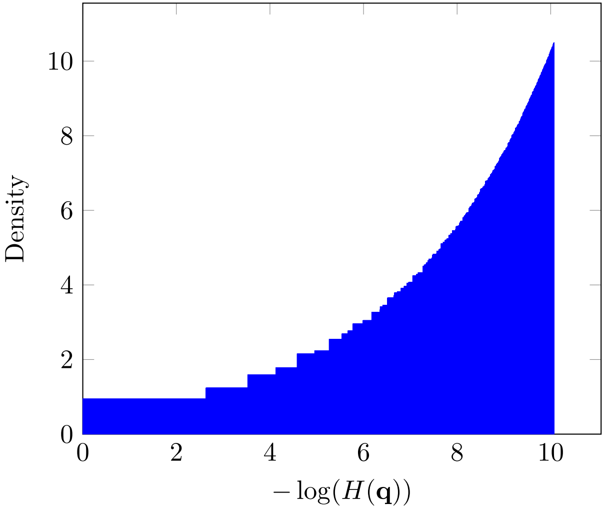
Table 1: The first 5 stairs

Cutoff	Height
1	$\frac{3}{\pi}$
$7-\sqrt{7^2-1}$	$\frac{2}{\sqrt{7^2-1}}$
$17-\sqrt{17^2-1}$	$\frac{6}{\sqrt{17^2-1}}$
$31-\sqrt{31^2-1}$	$\frac{6}{\sqrt{31^2-1}}$
$49-\sqrt{49^2-1}$	$\frac{6+12}{\sqrt{49^2-1}}$

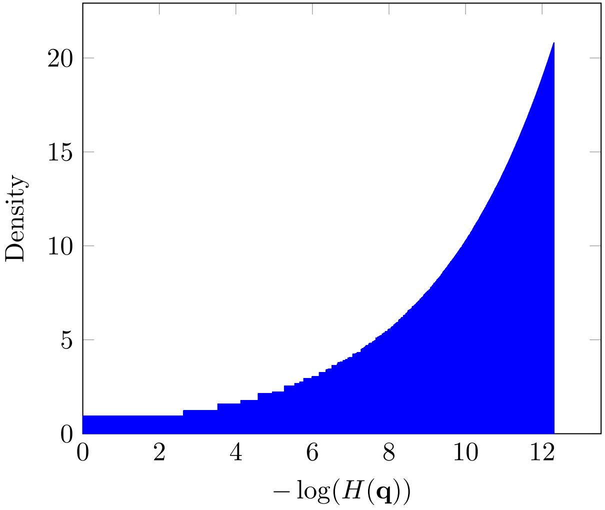
First 40 stairs



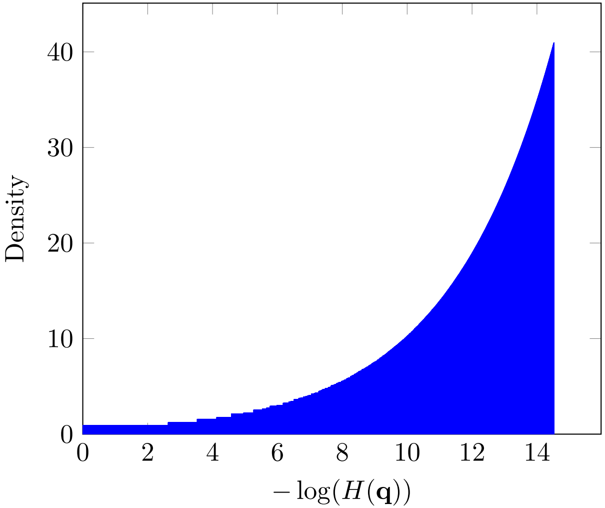
First 400 stairs



First 4000 stairs



First 40000 stairs



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Theorem (R., 2022)

Spikes appear in the histogram of $\mathrm{RMC}(n)$ *for each prime* $p \mid n$ *with* $p \leq \sqrt{n}$*. Primes close to* $\sqrt{n}$ *give rise to a small number of tall spikes, whereas primes close to* 1 *give rise to a large number of short spikes.*

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- $n = 33920039$ is prime (no spikes);

Main results

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Spikes appear in the histogram of $\mathrm{RMC}(n)$ *for each prime* $p \mid n$ *with* $p \leq \sqrt{n}$*. Primes close to* $\sqrt{n}$ *give rise to a small number of tall spikes, whereas primes close to* 1 *give rise to a large number of short spikes.*

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- $n = 33920039$ is prime (no spikes);
- $n = 42728555 = 5 \cdot 101 \cdot 211 \cdot 401$ gives a variety of spikes.
- Spikes do not affect the previous theorem, as this deals with fixed bin sizes, and the effect of spikes are washed away.

$32109677 = 19 \cdot 251 \cdot 6733$

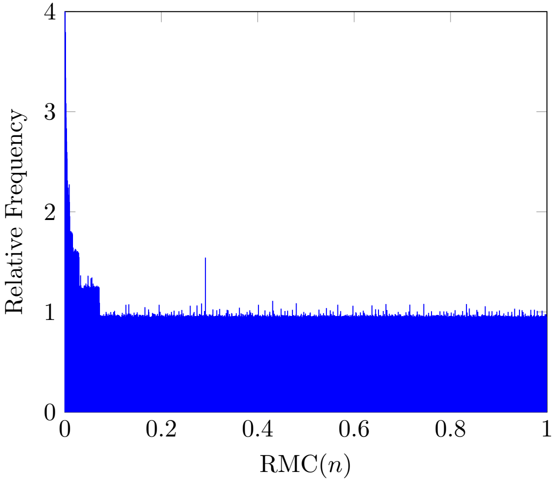

Figure 11: Histogram for $n = 32109677$; 8482324 data points in 2000 bins.

Tangency to the outer circle

Corollary

Let $n \in \mathbb{Z}^+$, and pick a random primitive integral Apollonian circle packing containing a circle of curvature n. Then the probability that this circle is tangent to the outer circle tends to $3/\pi$ as $n \to \infty$.

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They fix a circle C in an Apollonian circle packing, and consider $\epsilon-$neighbourhoods of the exterior of C. It is shown that the proportion of points in the neighbourhood that lie in a circle tangent to C tends to $\frac{3}{\pi}$ as $\epsilon \to 0$.

Parameter space, revisited

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- We can restrict to a fundamental domain for $\mathrm{PGL}(2, \mathbb{Z})$, U_{PGL}!
- Picking a random packing containing the circle can be simulated by picking a random point in U_{PGL} with respect to the hyperbolic metric.

Fundamental domain for $\mathrm{PGL}(2, \mathbb{Z})$

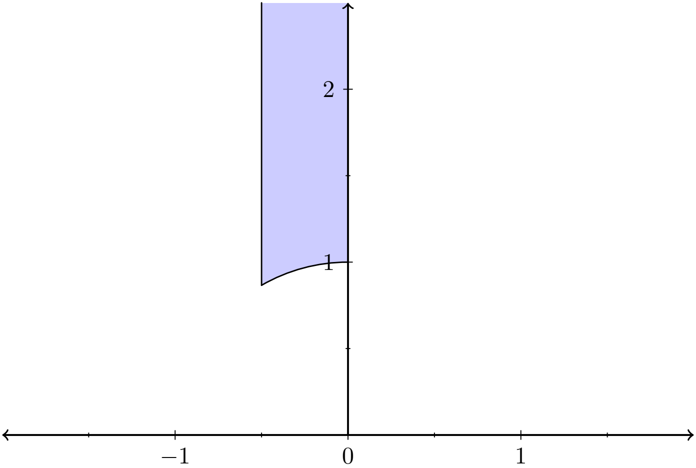

Figure 12: U_{PGL}

Depth elements

Definition

Let $\mathbf{q}$ be a Descartes quadruple that generates a bounded or half-plane packing. The depth element associated to $\mathbf{q}$ is the smallest word W in S_1, S_2, S_3, S_4 such that $W\mathbf{q}$ contains a circle of non-positive curvature.

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In the strip packing, we associate two depth elements, one per curvature zero circle.

If the depth element of $\mathbf{q}$ is Id, then we add a subscript to denote the index of the non-positive curvature.

Depth elements - example

$$\begin{aligned}\mathbf{q} =& (3694, 1963, 154, 111)\\ S_1\mathbf{q} =& (762, 1963, 154, 111)\\ S_2S_1\mathbf{q} =& (762, 91, 154, 111)\\ S_1S_2S_1\mathbf{q} =& (-50, 91, 154, 111)\end{aligned}$$

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Therefore the depth element of $\mathbf{q}$ is $S_1S_2S_1$.

Depth elements - example

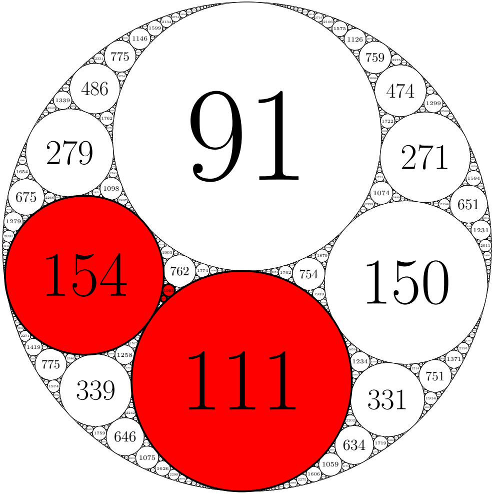

Figure 13: $\mathbf{q} = (3694, 1963, 154, 111)$

Depth elements - example

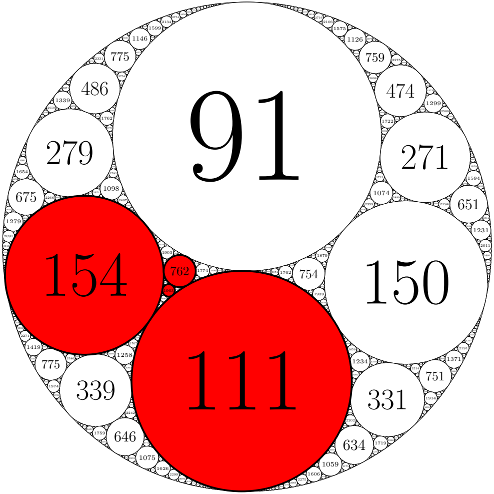

Figure 13: $S_1\mathbf{q} = (762, 1963, 154, 111)$

Depth elements - example

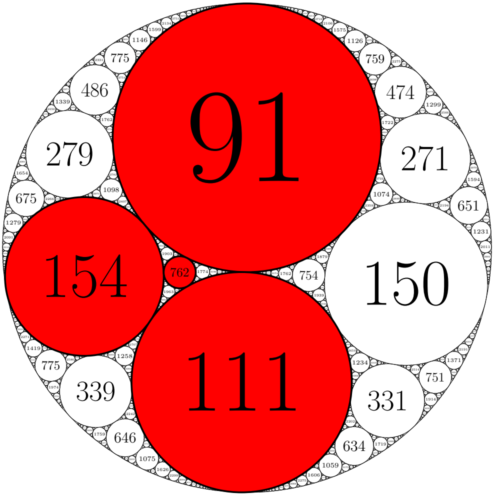

Figure 13: $S_2 S_1 \mathbf{q} = (762, 91, 154, 111)$

Depth elements - example

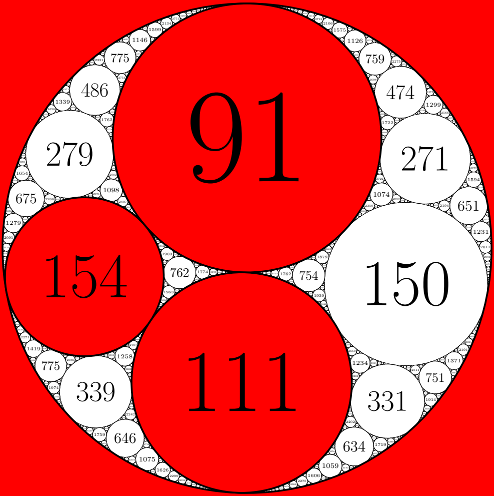

Figure 14: $S_1 S_2 S_1 \mathbf{q} = (-50, 91, 154, 111)$

Depth elements in $\mathbb{C}$

Fix $n \in \mathbb{R}^+$, and fix a depth element W. What is the distribution of points in $\mathbb{H}$ corresponding to Descartes quadruples with depth element W?

Depth elements in $\mathbb{C}$

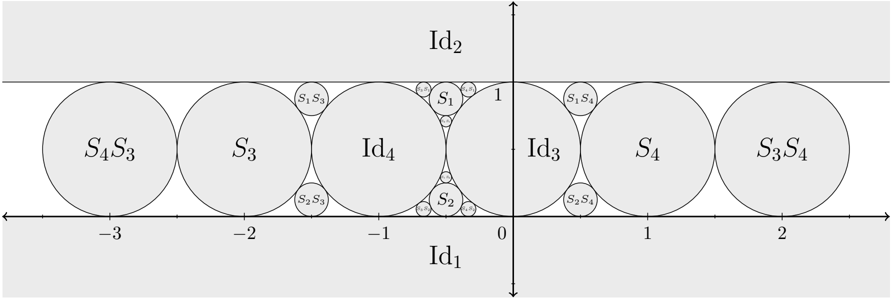

Figure 15: Depth circles with $\delta(\mathbf{q}) \leq 2$, labelled by depth element.

Depth elements in $\mathbb{C}$

This is the strip packing! Call the corresponding regions *depth circles*.

t

Given a depth element W, let t be the product of the curvature and $y-$coordinate of the centre of its depth circle. Then $t \in \{1, 7, 17, 31, 49, \ldots\} \subseteq \mathbb{Z}$.

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Proposition

If the depth circle does not touch the x-axis, then $H(\mathbf{q})$ is uniformly distributed in $[0, t - \sqrt{t^2 - 1}]$ for p_q uniformly distributed with respect to the hyperbolic metric (in the depth circle).

Intersection with the fundamental domain

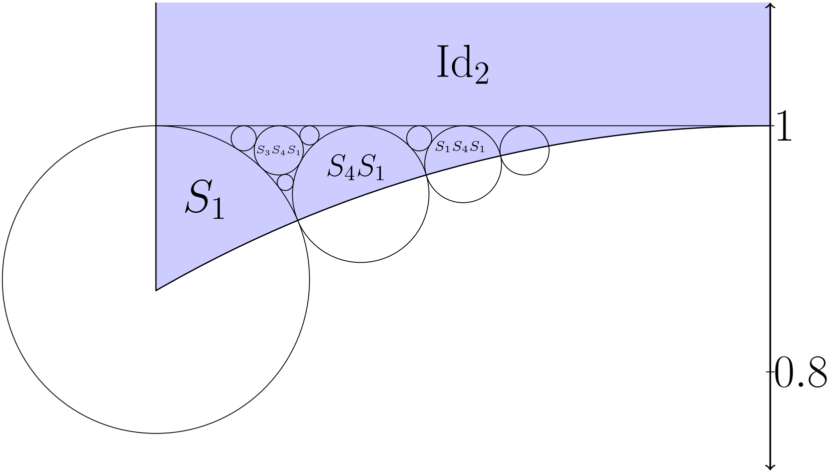

Figure 15: $D_4 \bigcap U_{\mathrm{PGL}}$.

Stairs, revisited

Table 2: The first 5 stairs

Depth element(s)	t	Cutoff	Height
Id_2	1	1	$\frac{3}{\pi}$
S_1	7	$7-\sqrt{7^2-1}$	$\frac{2}{\sqrt{7^2-1}}$
S_4S_1	17	$17-\sqrt{17^2-1}$	$\frac{6}{\sqrt{17^2-1}}$
$S_1S_4S_1$	31	$31-\sqrt{31^2-1}$	$\frac{6}{\sqrt{31^2-1}}$
$S_3S_4S_1, S_4S_1S_4S_1$	49	$49-\sqrt{49^2-1}$	$\frac{6+12}{\sqrt{49^2-1}}$

Integral theory

This proves the results when we pick a random (with respect to the hyperbolic metric) point from the fundamental domain.

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A theorem of Duke ([Duk88]) says that Heegner points equidistribute as $n \to \infty$.

Tangency number

Definition

Let $c_1, c_2 \in \mathbb{Z}$. The tangency number of c_1, c_2, denoted $T(c_1, c_2)$, is the number of primitive integral circle packings with tangent circles of curvatures c_1 and c_2.

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Let $n \in \mathbb{Z}^+$ and let $c \in \mathbb{Z}$ satisfy $0 \leq c < n$. A spike will occur for RMC(n) at c/n if $T(n, -c)$ is much larger than expected.

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We can translate this problem to quadratic forms!

Size of the tangency number

Lemma

The tangency number $T(n, -c)$ *is equal to the number of integral solutions to* $B^2 - 4(n-c)C = -4n^2$ *with* $\gcd(n-c, B, C) = 1$ *and* $0 \leq B \leq n-c$.

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- This is essentially proportional the number of square roots of $-n^2$ modulo $n-c$.

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- We can do this count for each $p^e \mid\mid n-c$, and multiply together.

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- This allows us to predict the locations and relative heights of the spikes.

Acknowledgments and References

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