Failure of the local-global conjecture in thin (semi)groups

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- Generalize to semigroups in $\mathrm{GL}(n, \mathbb{Z})$

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- $\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ is NOT thin.
- $\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ IS thin.

Examples

- How about $\Gamma = \left\langle \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 0 & -1 \\ -5 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix} \right\rangle$?

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- $\mathrm{Zcl}(\Gamma) = \mathrm{SL}(3)$;
- Unknown if Γ is thin or not.
- In general, it is a difficult question to determine if a group is thin or not.

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Thin orbits

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- Let $v \in \mathbb{Z}^n$ be a primitive vector.
- Let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional.
- What can we say about $L(\Gamma \cdot v) \subseteq \mathbb{Z}$?

Example: Zaremba

Definition

The continued fraction expansion of a real number $\alpha \in \mathbb{R}$ is $[a_0; a_1, a_2, a_3, \ldots]$, where

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

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If $\alpha \in \mathbb{Q}$, it has a finite expansion.

Zaremba: motivation

- Zaremba (60s and 70s): generate pseudo-random numbers via

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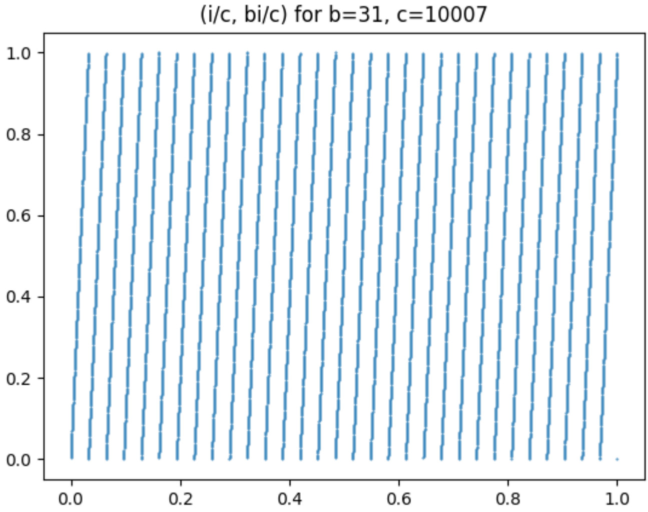
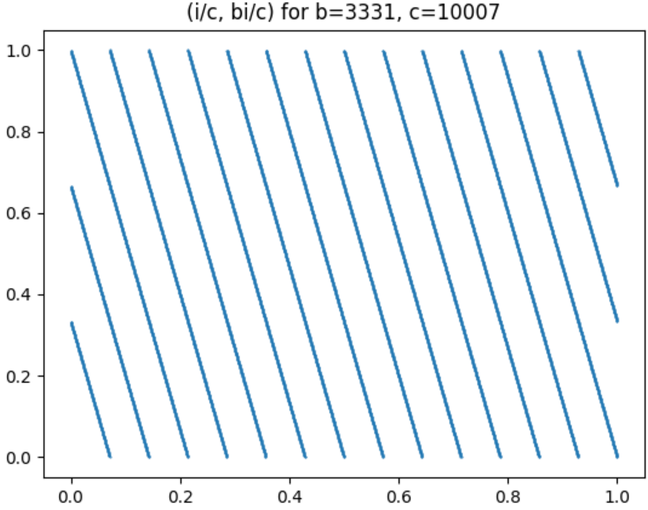
- If partial quotients of b/c are all small, the map seems good.
- Conjecture: given c, there exists coprime b so that b/c has only small partial quotients.

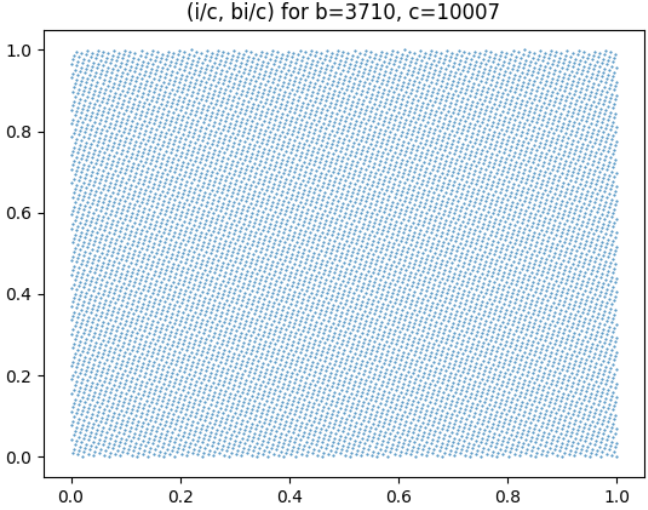
$\frac{31}{10007} = [0; 322, 1, 4, 6]$



$\frac{3331}{10007} = [0; 3, 237, 1, 13]$



$\frac{3710}{10007} = [0; 2, 1, 2, 3, 3, 2, 2, 3, 1, 1, 2]$



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For every positive integer n, there exists a coprime integer m such that all the partial quotients of $\frac{m}{n}$ are bounded by A, where A is an absolute constant.

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- Best known result: if $A = 5$, a density 1 set of denominators appear (Bourgain—Kontorovich, Huang).

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- $D_{\mathcal{A}} = L(\Gamma \cdot v)$ where $L\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = y$ and $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

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Conjecture (Hensley)

If $\delta_{\mathcal{A}} > 1/2$, then $D_{\mathcal{A}}$ contains all but finitely many positive integers.

Hausdorff dimensions

- $\mathcal{A} = \{1, 2, 3, 4, 5\}$: $\delta_{\mathcal{A}} \approx 0.8368$;
- $\mathcal{A} = \{1, 2, 3, 4\}$: $\delta_{\mathcal{A}} \approx 0.7889$;
- $\mathcal{A} = \{1, 2, 3\}$: $\delta_{\mathcal{A}} \approx 0.7057$;
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- $\mathcal{A} = \{2, 4, 6, 8, 10\}$: $\delta_{\mathcal{A}} \approx 0.5174$;
- Bourgain-Kontorovich: this last alphabet misses 3 (mod 4), disproving Hensley's conjecture.

Even partial quotients

$$\frac{x}{y} \to \frac{y}{x + 2ky}$$

- $x \equiv 0 \pmod 2$, $y \equiv 1 \pmod 4 \to x_{\text{new}} \equiv 1 \pmod 4$, $y_{\text{new}} \equiv 0 \pmod 2$

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- Kontorovich: more precise conjecture, incorporating a linear functional $L(x/y) = ax + by$ and an asymptotic count of the multiplicity.

Main Theorem

Theorem (R.-Stange)

Consider all rational numbers q which have a continued fraction of the form

$$q = [0; a_1, a_2, \ldots, a_k, b, 1, 2]$$

where $a_k \in \{4, 8, 12, 16, \ldots\} = 4\mathbb{Z}^+$ *and* $b \in \mathbb{Z}^+$. *Then no denominator of q is a perfect square.*

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Computed up to 2×10^{13}, last missing non-square is $7968219670470 \approx 7.9 \cdot 10^{12}$.

Consequences

- The denominator of
$$q = [0; a_1, a_2, \ldots, a_k, 1, 1, 2]$$
is $L(x, y) = 3x + 5y$, where
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- Taking $a_i \in \mathcal{A} = \{4, 8, 12, \ldots, 128\}$ guarantees $\delta_{\mathcal{A}} > 1/2$.
- Disproves the conjecture of Kontorovich

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- Quadratic reciprocity: Kronecker symbol preserved.
- Thus, denominator never a square!
- The “tail” of $[1, 1, 2]$: start the orbit at $\frac{3}{5}$ to have -1 Kronecker symbol.

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- Considering pure denominators: must start orbit at $\frac{0}{1}$, which has $\left(\frac{0}{1}\right) = 1$.
- Unclear whether an obstruction will exist for a general linear functional.

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If $x, y \in \mathbb{Z}^+$, y is odd, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Psi$, then

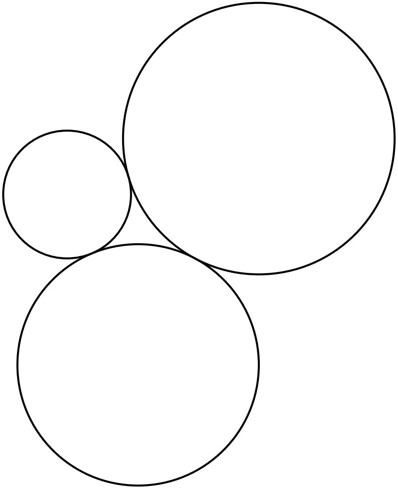
$$\left(\frac{ax+by}{cx+dy}\right) = \left(\frac{x}{y}\right).$$

Apollonian circle packings

Theorem (Apollonius)

Let three mutually tangent circles be drawn in the plane. Then there are exactly two more circles that can be drawn that are mutually tangent to the original three circles.

Example 1



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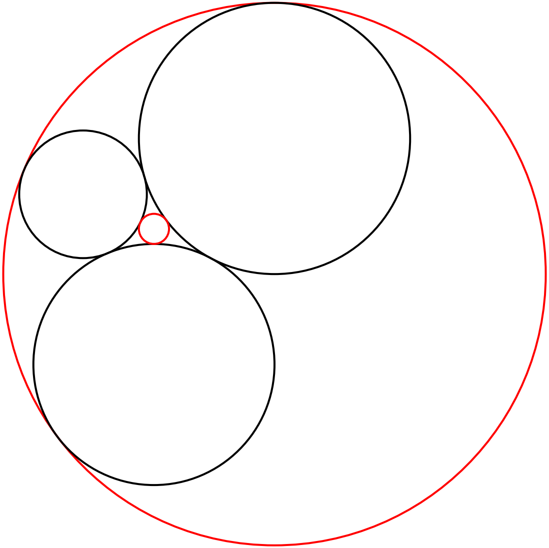
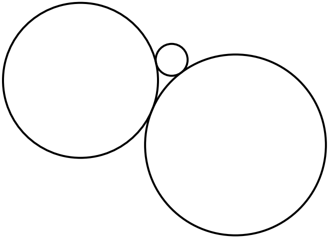


Example 2



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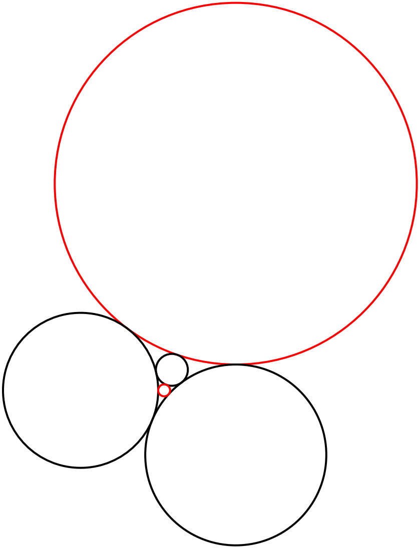

Apollonian circle packing: repeat!

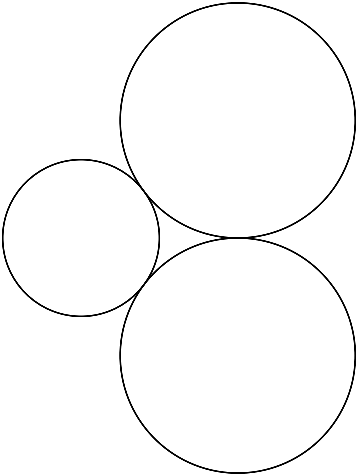

Figure 1: Generation 0

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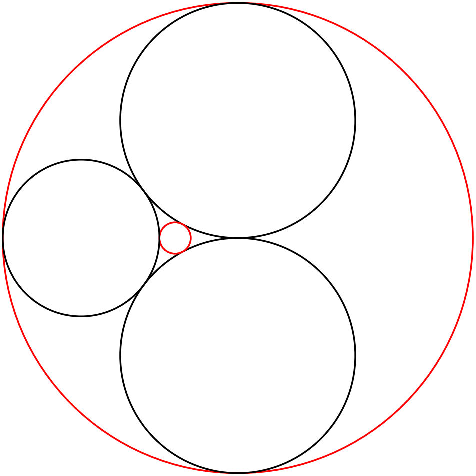

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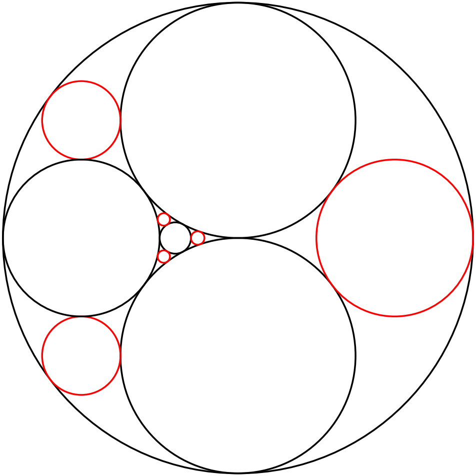

Figure 1: Generation 2

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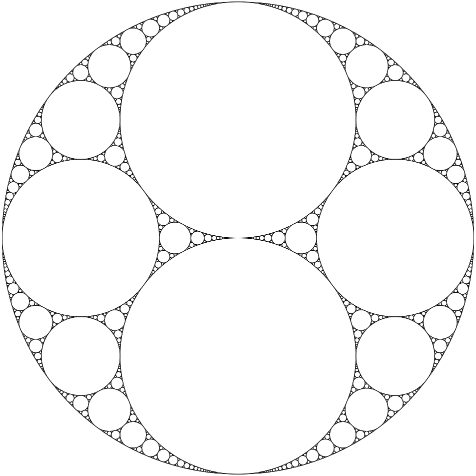

Figure 1: Radius $\geq \frac{1}{500}$.

Descartes equation

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Theorem (Descartes, Princess Elisabeth, 1643)

Let four mutually tangent circles have curvatures a, b, c, d. Then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Descartes equation

Definition

The curvature of a circle of radius r is $\frac{1}{r}$.

Theorem (Descartes, Princess Elisabeth, 1643)

Let four mutually tangent circles have curvatures a, b, c, d. Then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Definition

We call (a, b, c, d) a Descartes quadruple.

Circle swaps

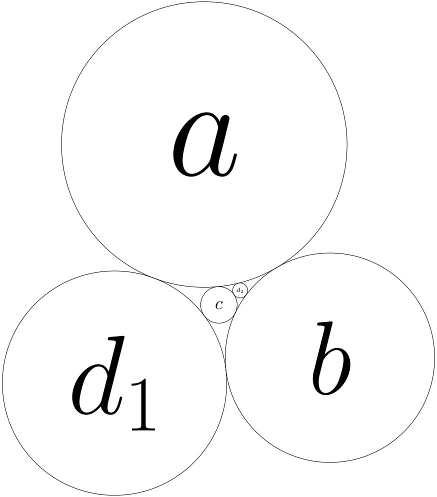

Figure 2: Swapping d_1 with d_2

Circle swaps - algebra

- $d = d_1$ and $d = d_2$ satisfy:

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

Circle swaps - algebra

- $d = d_1$ and $d = d_2$ satisfy:

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

- Rearrange:

$$d^2 - 2(a + b + c)d + 2(a^2 + b^2 + c^2) - (a + b + c)^2 = 0$$

Circle swaps - algebra

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- Rearrange:

$$d^2 - 2(a + b + c)d + 2(a^2 + b^2 + c^2) - (a + b + c)^2 = 0$$

- Vieta jumping:

$$d_1 + d_2 = 2a + 2b + 2c$$

Example

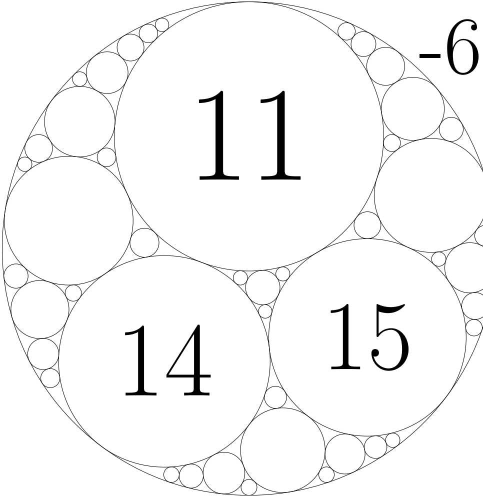

Figure 3: Quadruple $(-6, 11, 14, 15)$

Example

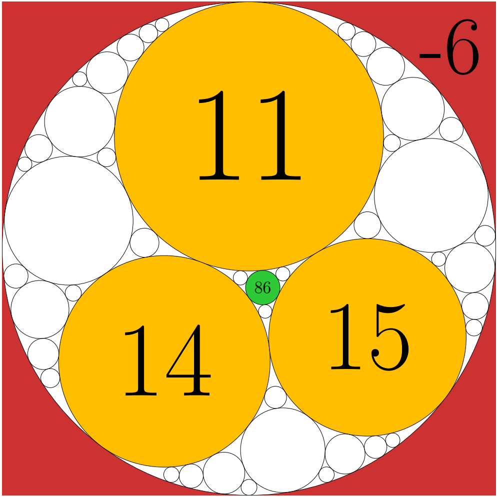

Figure 4: $2(11 + 14 + 15) - (-6) = 86$

Example

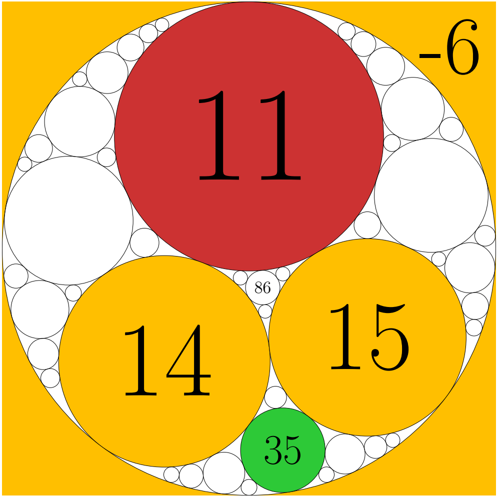

Figure 5: $2(-6+14+15)-11=35$

Example

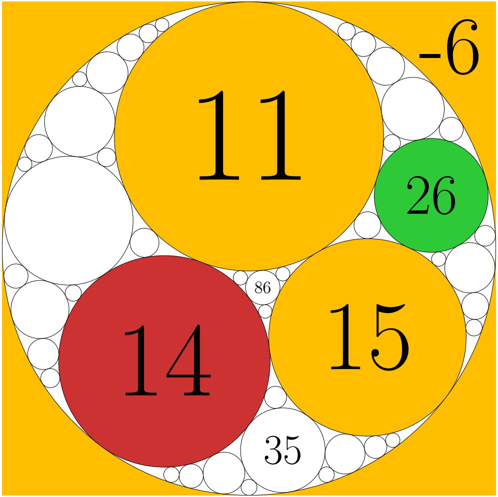

Figure 6: $2(-6 + 11 + 15) - 14 = 26$

Example

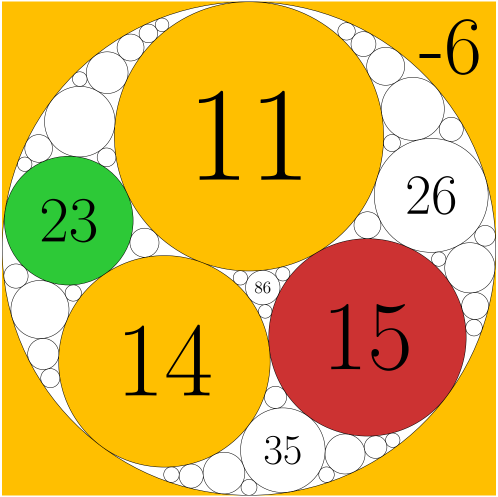

Figure 7: $2(-6+11+14)-15=23$

Example

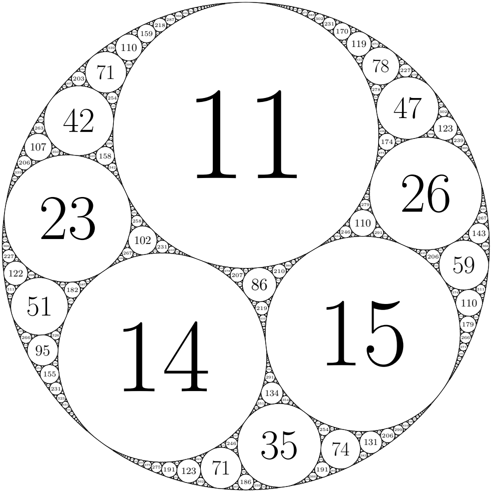

Figure 8: Repeat!

Integral theory

- If $a, b, c, d \in \mathbb{Z}$, then all curvatures are integers.

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Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Circle packings as thin groups

- The Apollonian group is

$$\Gamma = \left\langle \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix} \right\rangle.$$

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- Let $v \in \mathbb{Z}^4$ be a primitive solution to the Descartes equation.
- The curvatures of circles appearing in the corresponding circle packing are the entries of $\Gamma \cdot v$.

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- Let $v \in \mathbb{Z}^4$ be a primitive solution to the Descartes equation.
- The curvatures of circles appearing in the corresponding circle packing are the entries of $\Gamma \cdot v$.
- This is a union of four thin group orbits.

Curvatures

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- At most one negative curvature can appear.
- Hausdorff dimension $\delta \approx 1.3057$, circles of curvature at most N grows like cN^{δ}.
- Congruence obstructions modulo 24.

Mod 3

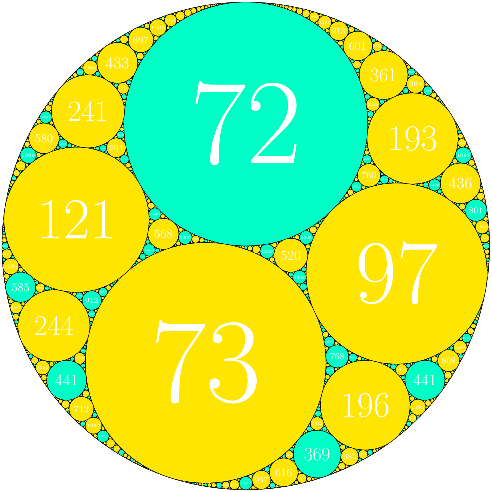

Figure 9: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Mod 8

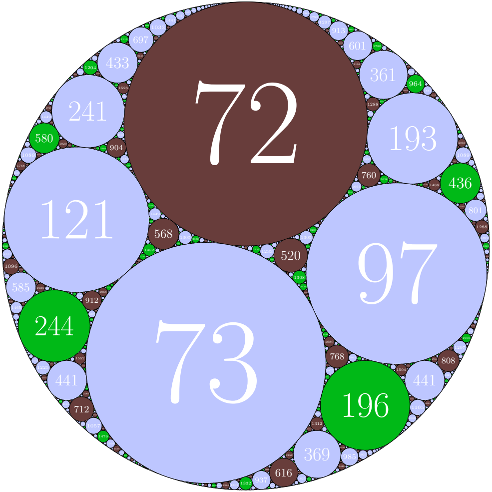

Figure 10: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Mod 24

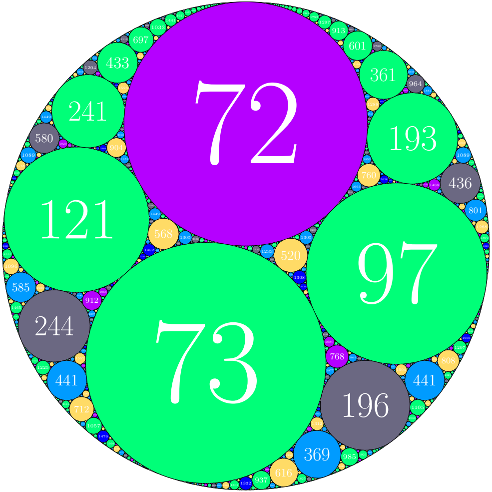

Figure 11: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Missing curvatures

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Call a positive curvature c *missing* in $\mathcal{A}$ if curvatures equivalent to $c \pmod{24}$ appear in $\mathcal{A}$ but c does not.

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Conjecture (Local-global conjecture: Graham-Lagarias-Mallows-Wilks-Yan and Fuchs-Sanden)

The number of missing curvatures is finite.

Local-global is false!

Theorem (Haag-Kertzer-R.-Stange)

There exist infinitely many primitive Apollonian circle packings for which the number of missing curvatures up to N is $\Omega(\sqrt{N})$. In particular, the local-global conjecture is false for these packings.

Local-global is false!

- More generally, we find families of the form

$$\begin{aligned} cx^2 &: \quad c \in \{1, 2, 3, 6\} \\ dx^4 &: \quad d \in \{1, 4, 9, 36\} \end{aligned}$$

which are entirely missing.

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- Such families are deemed *reciprocity obstructions*.
- Given a packing, we classify which (if any) reciprocity obstructions are present.
- Extensive computation suggests this list is complete.

Quadratic and quartic obstructions

Type	Quadratic	Quartic	L-G false	L-G open
$(6, 1, 1, 1)$				$0, 1, 4, 9, 12, 16$
$(6, 1, 1, -1)$		$n^4, 4n^4,$ $9n^4, 36n^4$	$0, 1, 4,$ $9, 12, 16$	
$(6, 1, -1)$	$n^2, 2n^2,$ $3n^2, 6n^2$		$0, 1, 4,$ $9, 12, 16$	
$(6, 5, 1)$	$2n^2, 3n^2$		$0, 8, 12$	$5, 20, 21$
$(6, 5, -1)$	$n^2, 6n^2$		$0, 12$	$5, 8, 20, 21$
$(6, 13, 1)$	$2n^2, 6n^2$		0	$4, 12, 13, 16, 21$
$(6, 13, -1)$	$n^2, 3n^2$		$0, 4, 12, 16$	$13, 21$
$(6, 17, 1, 1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, 1, -1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, -1)$	$n^2, 2n^2$		$0, 8, 9, 12$	$17, 20$
$(8, 7, 1)$	$3n^2, 6n^2$		$3, 6$	$7, 10, 15, 18, 19, 22$
$(8, 7, -1)$	$2n^2$		18	$3, 6, 7, 10, 15, 19, 22$
$(8, 11, 1)$				$2, 3, 6, 11, 14, 15, 18, 23$
$(8, 11, -1)$	$2n^2, 3n^2, 6n^2$		$2, 3, 6, 18$	$11, 14, 15, 23$

Example

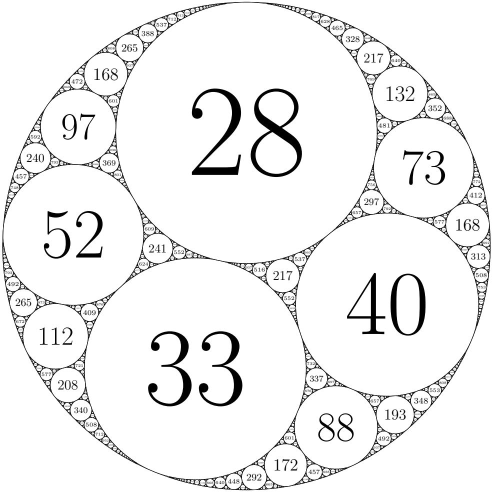


Figure 12: Missing $x^2, 2x^2, 3x^2, 6x^2$

Example

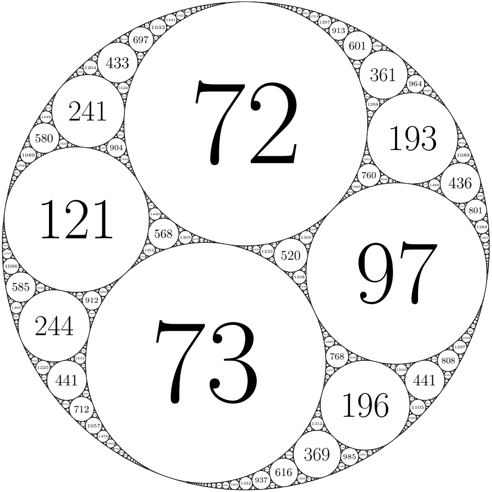


Figure 13: Missing $x^4, 4x^4, 9x^4, 36x^4$

Example

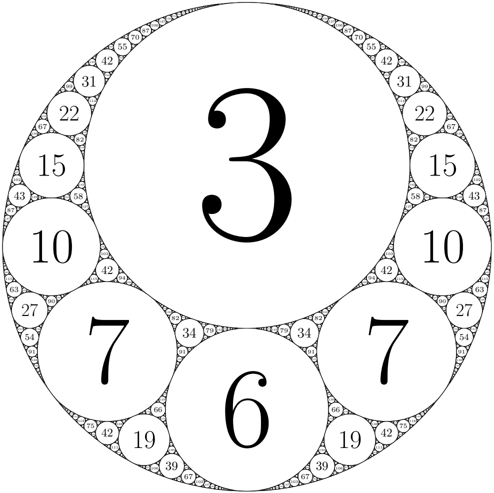


Figure 14: Missing $2x^2$

Example

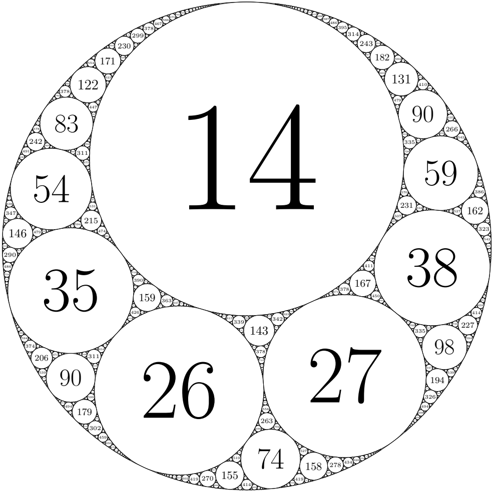


Figure 15: No reciprocity obstructions. Local-global may still be true!

Fix a packing

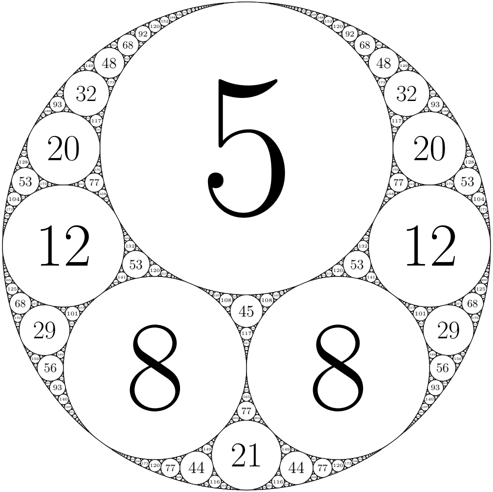


Figure 16: $(-3, 5, 8, 8)$

Fix a circle

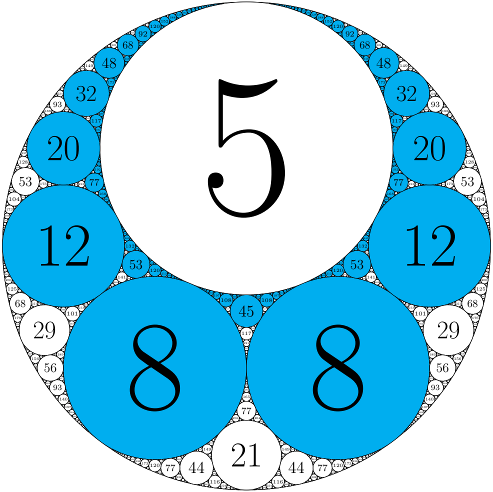
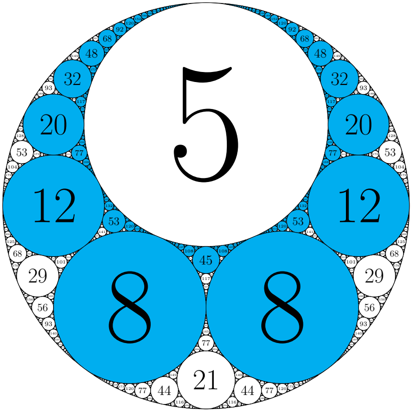


Figure 17: Circles tangent to curvature 5 circle

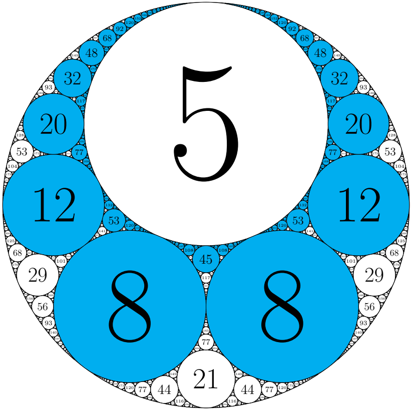
Curvatures of circles tangent to a fixed circle



Possible curvatures:

$-3, 8, 12, 20, 32, 45, 48, 53, 68, 77, \ldots$

Curvatures of circles tangent to a fixed circle



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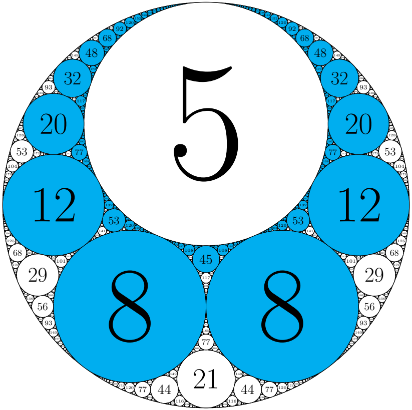
Consider the quadratic function

$$f(x, y) = 13x^2 + 24xy + 13y^2 - 5$$

Curvatures of circles tangent to a fixed circle



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$$-3, 8, 12, 20, 32, 45, 48, 53, 68, 77, \ldots$$

Consider the quadratic function

$$f(x, y) = 13x^2 + 24xy + 13y^2 - 5$$

Note that

$$\begin{aligned} -3 &= f(1, -1), \quad 8 = f(1, 0), \\ 12 &= f(2, -1), \quad 20 = f(3, -2), \\ 32 &= f(4, -3), \quad 45 = f(1, 1), \ldots \end{aligned}$$

Tangent circles

Observation of Sarnak:

$$\{\text{values of } f(x,y) \text{ with } x,y \text{ coprime integers}\} = \{\text{curvatures of tangent circles}\}$$

Tangent circles

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In general, from (a,b,c,d), to the first circle we associate

$$(a+b)x^2 + (a+b+c-d)xy + (a+c)y^2 - a.$$

The unshifted quadratic form has discriminant $-4a^2$.

Tangent circles

In our case,

$$f(x,y) \equiv 13x^2 + 24xy + 13y^2 - 5 \equiv 3x^2 - 6xy + 3y^2 \equiv 3(x-y)^2 \pmod 5$$

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If a circle of curvature c is tangent to the curvature 5 circle and $\gcd(c, 5) = 1$, then

$$\left(\frac{c}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

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If a circle of curvature c is tangent to the curvature 5 circle and $\gcd(c, 5) = 1$, then

$$\left(\frac{c}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

In particular, it cannot be square!

General case

Definition

Let $\mathcal{C}$ be a circle of curvature a, and let a tangent circle have curvature b with $\gcd(a, b) = 1$. Define

$$\chi_2(\mathcal{C}) = \left(\frac{b}{a}\right) \in \{-1, 1\}.$$

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$$\chi_2(\mathcal{C}) = \left(\frac{b}{a}\right) \in \{-1, 1\}.$$

A similar computation shows that this is well defined in the packing $(-3, 5, 8, 8)$.

Quadratic reciprocity

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent with coprime curvatures a, b, then

$$\chi_2(\mathcal{C}_1) = \left(\frac{b}{a}\right) = \left(\frac{a}{b}\right) = \chi_2(\mathcal{C}_2),$$

as one of them is $1 \pmod 4$.

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Consider the following graph:

- vertices = circles;
- edges between tangent circles of coprime curvature.

This graph is connected! Thus χ_2 is constant across the packing.

Sample path

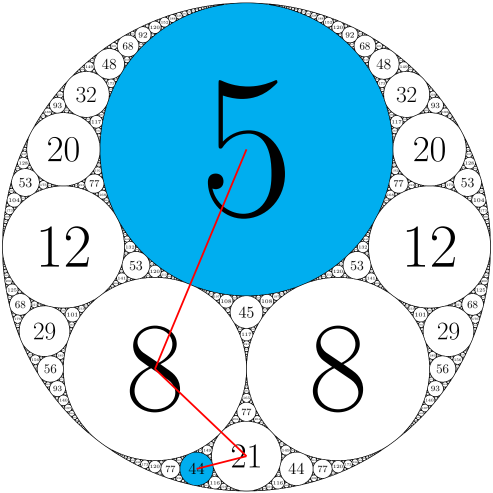

Figure 18: One coprime path

Consequences

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Thus $\chi_2 = -1$ for the entire packing!

If a curvature was a perfect square, then

$$\chi_2(\mathcal{C}) = \left(\frac{b}{n^2}\right) = 1,$$

contradiction.

New conjecture

Definition

Call a curvature that is not ruled out by congruence conditions or one of our reciprocity obstructions *sporadic*.

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Conjecture

There are finitely many sporadic curvatures.

In other words, we have found the only exceptions to the local-global conjecture.

Back to thin orbits

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be thin.
- Let $v \in \mathbb{Z}^n$ be a primitive vector.
- Let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional.
- What can we say about $L(\Gamma \cdot v) \subseteq \mathbb{Z}$?

$L(\Gamma \cdot v)$ standard obstructions

- **Inherited**: obstructions may exist for an algebraic set containing Γ.

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- **Definiteness**: $L(\Gamma \cdot v)$ may only represent numbers above/below a certain cutoff.

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- **Congruence**: an integer n is *admissible* if integers equivalent to $n \pmod{d}$ occur in $L(\Gamma \cdot v)$ for all $d \in \mathbb{Z}^{+}$.
- Congruence obstructions boil down to a single modulus.

Local-global conjecture

- These obstructions *completely* describe $L(\Gamma \cdot v)$.

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- Typical statement: "For all sufficiently large integers n such that $n \pmod{N} \notin X$, $n \in L(\Gamma \cdot v)$."
- Our two examples disprove this philosophy by adding in *reciprocity obstructions*.

What is a reciprocity obstruction?

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- Individual orbits may or may not have a corresponding congruence obstruction.
- Should be an analogous definition for reciprocity obstructions.
- Required to *prove* that a group has no reciprocity obstructions.

What is a reciprocity obstruction?

- Congruence obstructions are a symptom of *strong approximation*, a property of the *group*.
- Individual orbits may or may not have a corresponding congruence obstruction.
- Should be an analogous definition for reciprocity obstructions.
- Required to *prove* that a group has no reciprocity obstructions.
- Need more examples!

Non-Apollonian circle packing

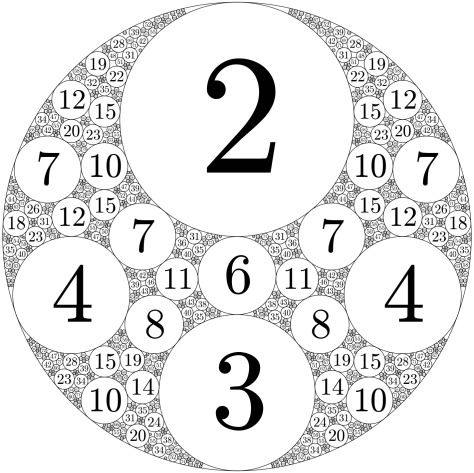

Figure 19: A non-Apollonian circle packing (Katherine E. Stange)