The not-so-local-global conjecture

James Rickards
Joint work with Summer Haag, Clyde Kertzer, Katherine E. Stange

CU Boulder

james.rickards@colorado.edu

10 October 2023

Apollonius of Perga

Theorem

Let three mutually tangent circles be drawn in the plane. Then there are exactly two more circles that can be drawn that are mutually tangent to the original three circles.

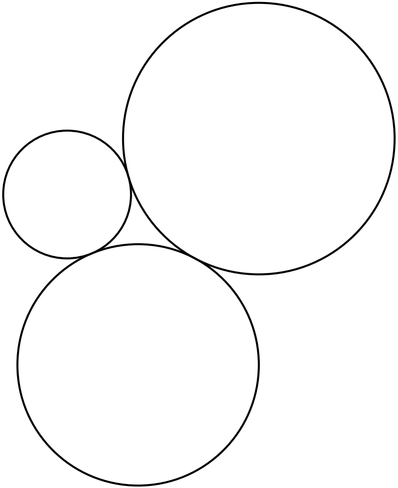
Apollonius of Perga

Theorem

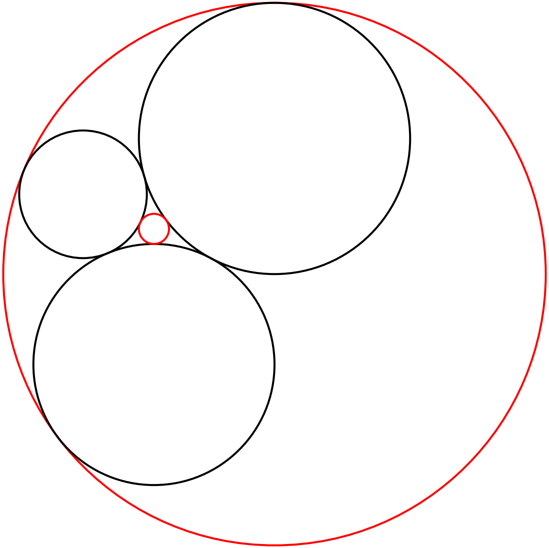
Let three mutually tangent circles be drawn in the plane. Then there are exactly two more circles that can be drawn that are mutually tangent to the original three circles.

Originally studied by Apollonius in the lost book "De tactionibus".

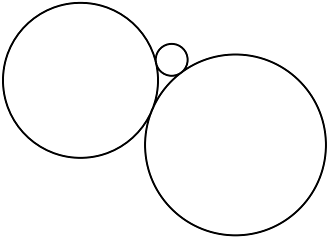
Example 1



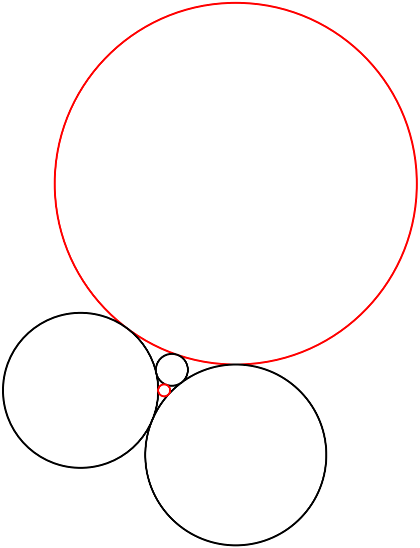
Example 1



Example 2



Example 2



Apollonian circle packing

- Start with three mutually tangent circles;

Apollonian circle packing

- Start with three mutually tangent circles;
- Draw the two circles tangent to all three;

Apollonian circle packing

- Start with three mutually tangent circles;
- Draw the two circles tangent to all three;
- This creates six more triples of mutually tangent circles. Repeat!

Apollonian circle packing - example

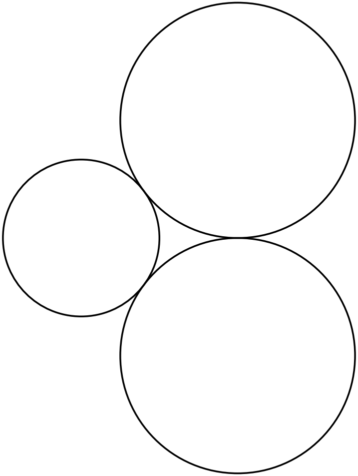


Figure 1: Generation 0

Apollonian circle packing - example

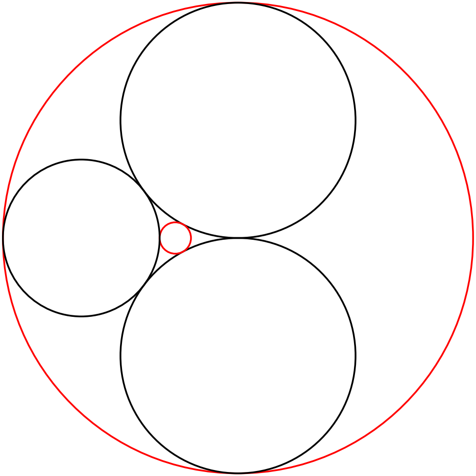


Figure 1: Generation 1

Apollonian circle packing - example

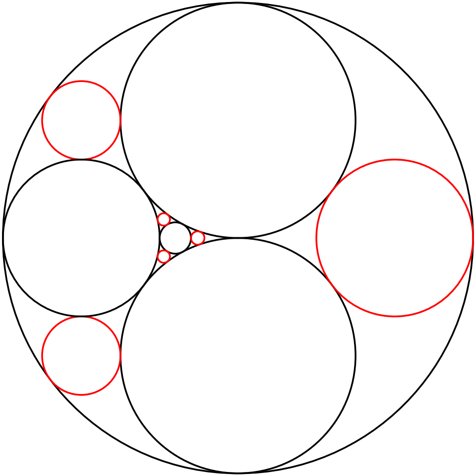


Figure 1: Generation 2

Apollonian circle packing - example

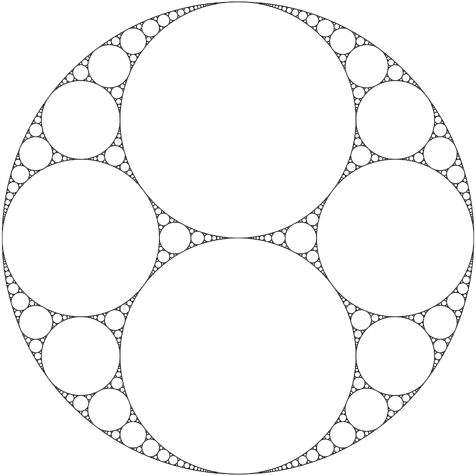


Figure 1: Radius $\geq \frac{1}{500}$.

Descartes equation

Definition

The curvature of a circle of radius r is $\frac{1}{r}$.

Descartes equation

Definition

The curvature of a circle of radius r is $\frac{1}{r}$.

Theorem (Descartes, Princess Elisabeth, 1643)

Let four mutually tangent circles have curvatures a, b, c, d. Then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Descartes equation

Definition

The curvature of a circle of radius r is $\frac{1}{r}$.

Theorem (Descartes, Princess Elisabeth, 1643)

Let four mutually tangent circles have curvatures a, b, c, d. Then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Definition

We call (a, b, c, d) a Descartes quadruple.

Types of packings

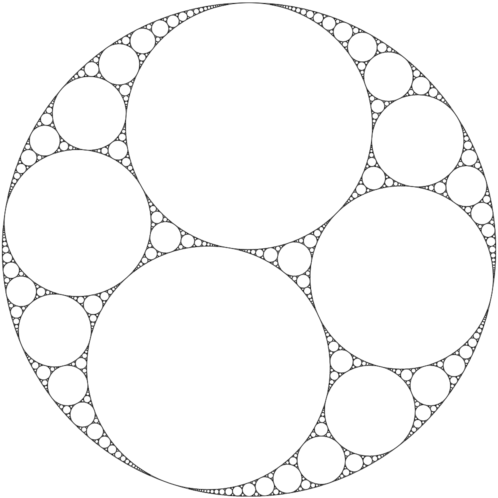


Figure 2: Bounded: $(-36, 72, 73, 97)$

Types of packings

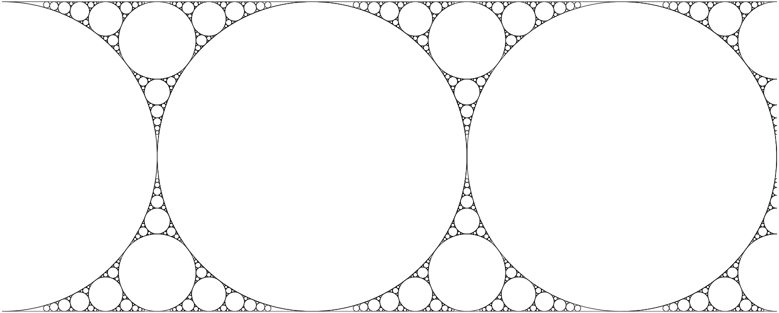


Figure 2: Strip: $(0, 0, 1, 1)$

Types of packings

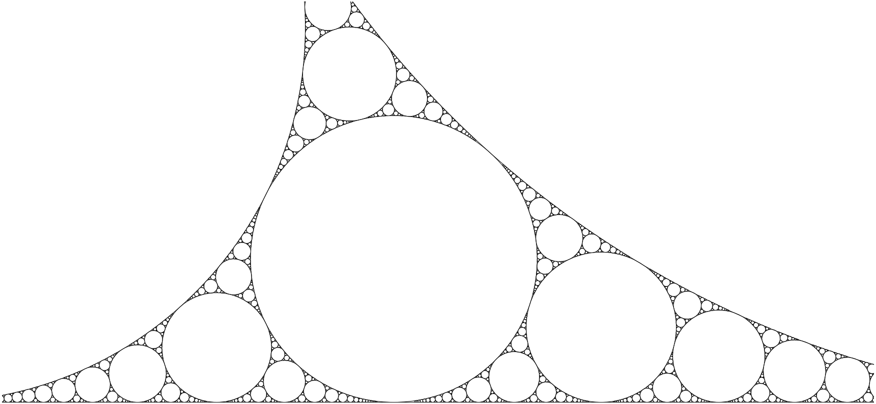


Figure 2: Half-plane: $(0, 1, \phi + 1, 3\phi + 2)$

Types of packings

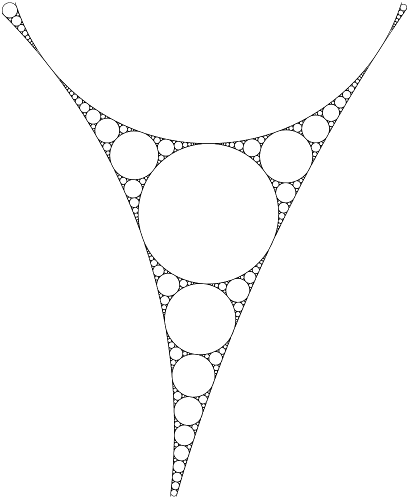


Figure 2: Full-plane: $(1, \phi - \sqrt{\phi}, (\phi - \sqrt{\phi})^2, (\phi - \sqrt{\phi})^3)$

Integral theory

- Let (a, b, c, d) be a Descartes quadruple, and assume that the other circle tangent to a, b, c has radius d'.

Integral theory

- Let (a, b, c, d) be a Descartes quadruple, and assume that the other circle tangent to a, b, c has radius d'.
- Descartes equation: $d^2 - 2(a+b+c)d + 2(a^2+b^2+c^2) - (a+b+c)^2 = 0$

Integral theory

- Let (a, b, c, d) be a Descartes quadruple, and assume that the other circle tangent to a, b, c has radius d'.
- Descartes equation: $d^2 - 2(a + b + c)d + 2(a^2 + b^2 + c^2) - (a + b + c)^2 = 0$
- Vieta jumping: $d' = 2a + 2b + 2c - d$

Integral theory

- Let (a, b, c, d) be a Descartes quadruple, and assume that the other circle tangent to a, b, c has radius d'.
- Descartes equation: $d^2 - 2(a+b+c)d + 2(a^2+b^2+c^2) - (a+b+c)^2 = 0$
- Vieta jumping: $d' = 2a + 2b + 2c - d$
- Therefore if $(a, b, c, d) \in \mathbb{Z}^4$, then $d' \in \mathbb{Z}$. In particular, all curvatures in the packing are integers.

Integral theory

- Let (a, b, c, d) be a Descartes quadruple, and assume that the other circle tangent to a, b, c has radius d'.
- Descartes equation: $d^2 - 2(a + b + c)d + 2(a^2 + b^2 + c^2) - (a + b + c)^2 = 0$
- Vieta jumping: $d' = 2a + 2b + 2c - d$
- Therefore if $(a, b, c, d) \in \mathbb{Z}^4$, then $d' \in \mathbb{Z}$. In particular, all curvatures in the packing are integers.
- We normally restrict to primitive packings, i.e. $\gcd(a, b, c, d) = 1$.

Example

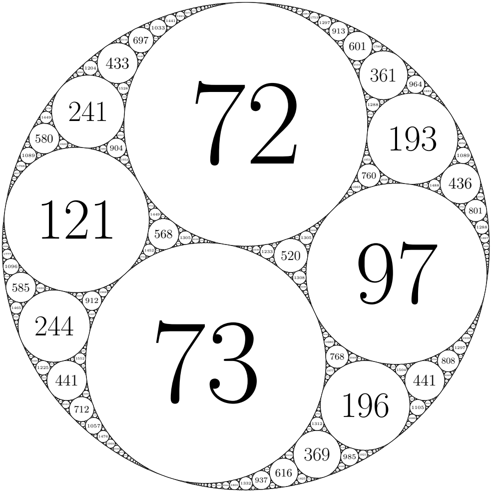


Figure 3: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Analytic constraints:

- Hausdorff dimension is $\delta \approx 1.3057$.
- The number of circles of curvature at most N is proportional to N^{δ}.
- The average multiplicity of a circle of size N is proportional to $\approx N^{0.3057} \to \infty$ as $N \to \infty$.

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Analytic constraints:

- Hausdorff dimension is $\delta \approx 1.3057$.
- The number of circles of curvature at most N is proportional to N^{δ}.
- The average multiplicity of a circle of size N is proportional to $\approx N^{0.3057} \to \infty$ as $N \to \infty$.

This suggests that every large enough curvature appears!

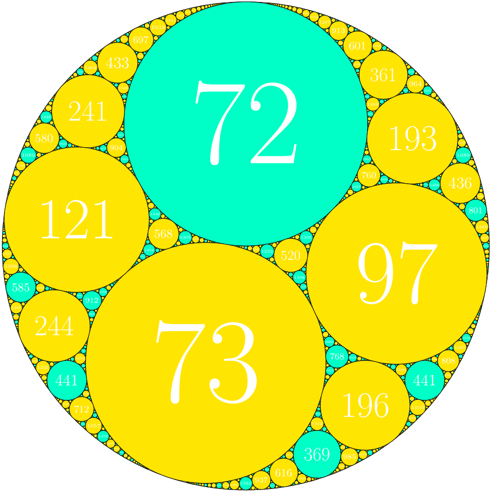


Figure 4: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Mod 8

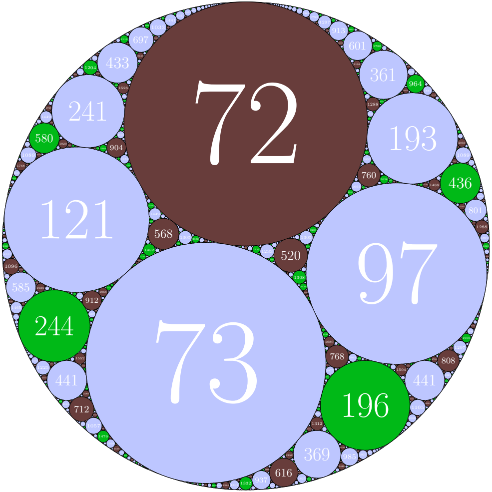


Figure 5: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Mod 24

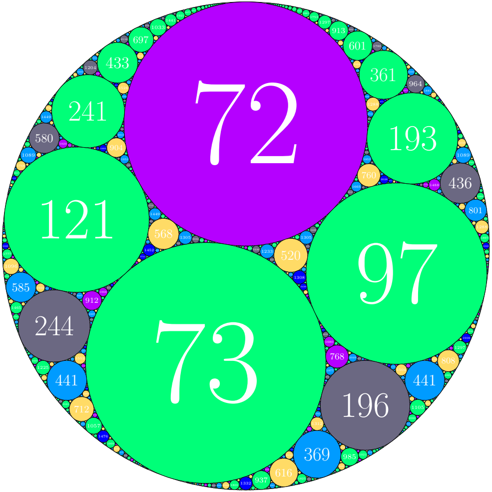


Figure 6: $(-36, 72, 73, 97)$, circles of curvature ≤ 20000

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Analytic constraints:

- None for sufficiently large integers.

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Analytic constraints:

- None for sufficiently large integers.

Algebraic constraints:

- Modulo 24, certain residue classes are missed.

Curvatures

Question

What curvatures appear in a fixed primitive Apollonian circle packing?

Analytic constraints:

- None for sufficiently large integers.

Algebraic constraints:

- Modulo 24, certain residue classes are missed.

Is this everything?

Missing curvatures

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Call a positive curvature c *missing* in $\mathcal{A}$ if curvatures equivalent to $c \pmod{24}$ appear in $\mathcal{A}$ but c does not.

Missing curvatures

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Call a positive curvature c *missing* in $\mathcal{A}$ if curvatures equivalent to $c \pmod{24}$ appear in $\mathcal{A}$ but c does not.

Conjecture (Local-Global Conjecture: Graham-Lagarias-Mallows-Wilks-Yan and Fuchs-Sanden, [GLM$^+$03, FS11])

The number of missing curvatures is finite.

Theoretical Evidence

Theorem (Density 1: Bourgain-Kontorovich, [BK14])

The number of missing curvatures up to N is at most $O(N^{1-\eta})$ for some effectively computable $\eta > 0$.

Theoretical Evidence

Theorem (Density 1: Bourgain-Kontorovich, [BK14])

The number of missing curvatures up to N is at most $O(N^{1-\eta})$ for some effectively computable $\eta > 0$.

Theorem (Fuchs, [Fuc10])

If a congruence obstruction appears, then it appears modulo 24.

Computational Evidence

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } (-1, 2, 2, 3)$$
$$5 \cdot 10^8 \text{ for } (-11, 21, 24, 28)$$

and observed that the multiplicity of a curvature was tending to increase.

Computational Evidence

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } (-1, 2, 2, 3)$$
$$5 \cdot 10^8 \text{ for } (-11, 21, 24, 28)$$

and observed that the multiplicity of a curvature was tending to increase.

For $(-11, 21, 24, 28)$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \pmod{24}$.

Computational Evidence

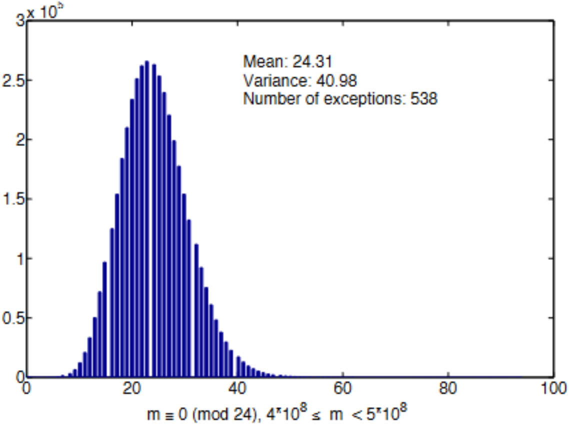


Figure 7: Missing curvatures 0 (mod 24) for $(-11, 21, 24, 28)$ (Fuchs-Sanden)

REU idea

- Fix a pair of curvatures, and study what packings contain them.

REU idea

- Fix a pair of curvatures, and study what packings contain them.
- One avenue: fix two curvatures, and ask whether there is at least one packing containing both curvatures or not.

REU idea

- Fix a pair of curvatures, and study what packings contain them.
- One avenue: fix two curvatures, and ask whether there is at least one packing containing both curvatures or not.
- For a pair of residue classes modulo 24 that can appear in the same packing, make a graph: white dot for at least one packing, black for none.

REU idea

- Fix a pair of curvatures, and study what packings contain them.
- One avenue: fix two curvatures, and ask whether there is at least one packing containing both curvatures or not.
- For a pair of residue classes modulo 24 that can appear in the same packing, make a graph: white dot for at least one packing, black for none.
- Local-global: finitely many black dots.

Typical graph

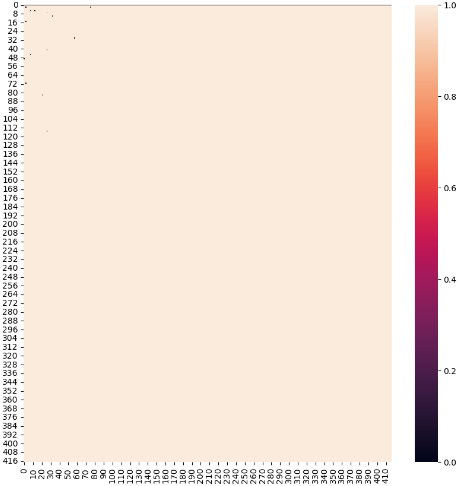


Figure 8: Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph

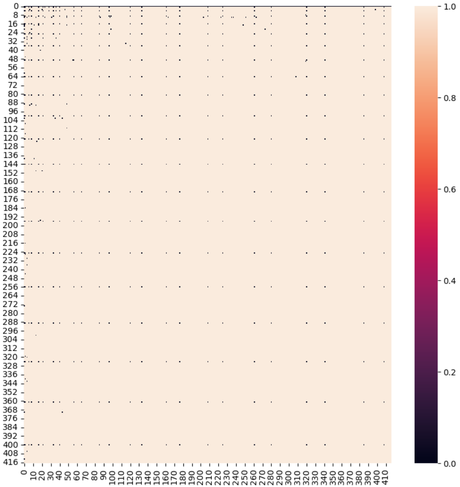


Figure 9: Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

$0 \pmod{24}$ and $8 \pmod{24}$

- There are tons of black dots, which appear not sporadically, but in lines.

$0 \pmod{24}$ and $8 \pmod{24}$

- There are tons of black dots, which appear not sporadically, but in lines.
- Uh oh, there must be a weird bug in the code!!

$0 \pmod{24}$ and $8 \pmod{24}$

- There are tons of black dots, which appear not sporadically, but in lines.
- Uh oh, there must be a weird bug in the code!!
- With a bit of help from OEIS, the data suggests that curvatures $24m^2$ and $8n^2$ with $3 \nmid n$ cannot occur in the same packing.

$0 \pmod{24}$ and $8 \pmod{24}$

- There are tons of black dots, which appear not sporadically, but in lines.
- Uh oh, there must be a weird bug in the code!!
- With a bit of help from OEIS, the data suggests that curvatures $24m^2$ and $8n^2$ with $3 \nmid n$ cannot occur in the same packing.
- This fact alone would disprove the local-global conjecture for any packing admitting both $0, 8 \pmod{24}$ curvatures.

Main result

Theorem (Haag-Kertzer-R.-Stange)

There exist infinitely many primitive Apollonian circle packings for which the number of missing curvatures up to N is $\Omega(\sqrt{N})$. In particular, the local-global conjecture is false for these packings.

Steps towards the precise result

- Study the residues modulo 24 of a primitive packing (the type);

Steps towards the precise result

- Study the residues modulo 24 of a primitive packing (the type);
- Introduce *reciprocity obstructions*, which can disprove the local-global conjecture;

Steps towards the precise result

- Study the residues modulo 24 of a primitive packing (the type);
- Introduce *reciprocity obstructions*, which can disprove the local-global conjecture;
- Define χ_2 and χ_4, whose values on a packing (alongside the type) determines the set of reciprocity obstructions.

Packing type

Proposition

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Let $R(\mathcal{A})$ be the set of residues modulo 24 of the curvatures in $\mathcal{A}$. Then $R(\mathcal{A})$ is one of six possible sets, labelled by a type as follows:

Type	$R(\mathcal{A})$
$(6,1)$	$0, 1, 4, 9, 12, 16$
$(6,5)$	$0, 5, 8, 12, 20, 21$
$(6,13)$	$0, 4, 12, 13, 16, 21$
$(6,17)$	$0, 8, 9, 12, 17, 20$
$(8,7)$	$3, 6, 7, 10, 15, 18, 19, 22$
$(8,11)$	$2, 3, 6, 11, 14, 15, 18, 23$

Reciprocity obstructions

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, let u and d be positive integers, and let $S_{d,u} := \{un^d : n \in \mathbb{Z}\}$. We say that the set $S_{d,u}$ forms a *reciprocity obstruction* to $\mathcal{A}$ if

- Infinitely many elements of $S_{d,u}$ are admissible in $\mathcal{A}$ modulo 24;
- No element of $S_{d,u}$ appears as a curvature in $\mathcal{A}$.

Reciprocity obstructions

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, let u and d be positive integers, and let $S_{d,u} := \{un^d : n \in \mathbb{Z}\}$. We say that the set $S_{d,u}$ forms a *reciprocity obstruction* to $\mathcal{A}$ if

- Infinitely many elements of $S_{d,u}$ are admissible in $\mathcal{A}$ modulo 24;
- No element of $S_{d,u}$ appears as a curvature in $\mathcal{A}$.

If $d = 2$ we call it a *quadratic obstruction*, and if $d = 4$ it is a *quartic obstruction*.

χ_2 and χ_4

There exists a function

$$\chi_2 : \{\text{circles in a primitive Apollonian circle packing}\} \to \{\pm 1\}$$

which relates to the possible curvatures of circles tangent to the input circle $\mathcal{C}$, and is constant across the packing containing $\mathcal{C}$.

χ_2 and χ_4

There exists a function

$$\chi_2 : \{\text{circles in a primitive Apollonian circle packing}\} \to \{\pm 1\}$$

which relates to the possible curvatures of circles tangent to the input circle $\mathcal{C}$, and is constant across the packing containing $\mathcal{C}$.
Furthermore, there exists a function

$$\chi_4 : \{\text{circles in a primitive Apollonian circle packing of type } (6,1) \text{ or } (6,17)\} \to \{1, i, -1, -i\}$$

that satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$, and is also constant across a packing.

χ_2 and χ_4

There exists a function

$$\chi_2 : \{\text{circles in a primitive Apollonian circle packing}\} \to \{\pm 1\}$$

which relates to the possible curvatures of circles tangent to the input circle $\mathcal{C}$, and is constant across the packing containing $\mathcal{C}$.
Furthermore, there exists a function

$$\chi_4 : \{\text{circles in a primitive Apollonian circle packing of type } (6,1) \text{ or } (6,17)\} \to \{1, i, -1, -i\}$$

that satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$, and is also constant across a packing.
The values of χ_2 and χ_4 determine the quadratic and quartic obstructions respectively.

χ_2 special case

Proposition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, and let (a, b) be a pair of curvatures of circles tangent to each other in $\mathcal{A}$ that also satisfies:

- *a is coprime to $6b$;*
- *if $\mathcal{A}$ is of type $(8, k)$, then $a \equiv 7 \pmod 8$.*

Then $\chi_2(\mathcal{A}) = \left(\frac{b}{a}\right)$.

χ_2 special case

Proposition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, and let (a, b) be a pair of curvatures of circles tangent to each other in $\mathcal{A}$ that also satisfies:

- *a is coprime to $6b$;*
- *if $\mathcal{A}$ is of type $(8, k)$, then $a \equiv 7 \pmod 8$.*

Then $\chi_2(\mathcal{A}) = \left(\frac{b}{a}\right)$.

The definition of χ_4 relies on a finer invariant using the quartic residue symbol for Gaussian integers.

Type, revisited

For a primitive Apollonian circle packing $\mathcal{A}$, the type can refer to any of the following:

- (x, k), as before;
- $(x, k, \chi_2(\mathcal{A}))$;
- $(x, k, \chi_2(\mathcal{A}), \chi_4(\mathcal{A}))$ if (x, k) is either $(6, 1)$ or $(6, 17)$.

Quadratic and quartic obstructions

Type	Quadratic	Quartic	L-G false	L-G open
$(6,1,1,1)$				$0,1,4,9,12,16$
$(6,1,1,-1)$		$n^4, 4n^4,$ $9n^4, 36n^4$	$0,1,4,$ $9,12,16$	
$(6,1,-1)$	$n^2, 2n^2,$ $3n^2, 6n^2$		$0,1,4,$ $9,12,16$	
$(6,5,1)$	$2n^2, 3n^2$		$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2, 6n^2$		$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2, 6n^2$		0	$4,12,13,16,21$
$(6,13,-1)$	$n^2, 3n^2$		$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2, 2n^2$		$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2, 6n^2$		$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$		18	$3,6,7,10,15,19,22$
$(8,11,1)$				$2,3,6,11,14,15,17,23$
$(8,11,-1)$	$2n^2, 3n^2, 6n^2$		$2,3,6,18$	$11,14,15,23$

Consequences

For packings of type $(6, 1, 1, -1)$ or $(6, 1, -1)$, the local-global conjecture is false for every single residue class. Examples:

$$(6, 1, 1, -1)\text{: } (-8, 12, 25, 25),\ (-12, 25, 25, 28),\ (-23, 48, 49, 52),\ \ldots$$
$$(6, 1, -1)\text{: } (-15, 28, 33, 40),\ (-20, 33, 52, 57),\ (-23, 40, 57, 60),\ \ldots$$

Consequences

For packings of type $(6,1,1,-1)$ or $(6,1,-1)$, the local-global conjecture is false for every single residue class. Examples:

$$(6,1,1,-1)\colon (-8,12,25,25), (-12,25,25,28), (-23,48,49,52), \ldots$$
$$(6,1,-1)\colon (-15,28,33,40), (-20,33,52,57), (-23,40,57,60), \ldots$$

For packings of type $(6,1,1,1)$ or $(8,11,1)$, the local-global conjecture may still be true in every residue class. Examples:

$$(6,1,1,1)\colon (0,0,1,1), (-12,16,49,49), (-20,36,49,49), \ldots$$
$$(8,11,1)\colon (-1,2,2,3), (-9,14,26,27), (-10,18,23,27), \ldots$$

Consequences

For packings of type $(6, 1, 1, -1)$ or $(6, 1, -1)$, the local-global conjecture is false for every single residue class. Examples:

$$(6,1,1,-1)\text{: } (-8,12,25,25), (-12,25,25,28), (-23,48,49,52), \ldots$$
$$(6,1,-1)\text{: } (-15,28,33,40), (-20,33,52,57), (-23,40,57,60), \ldots$$

For packings of type $(6, 1, 1, 1)$ or $(8, 11, 1)$, the local-global conjecture may still be true in every residue class. Examples:

$$(6,1,1,1)\text{: } (0,0,1,1), (-12,16,49,49), (-20,36,49,49), \ldots$$
$$(8,11,1)\text{: } (-1,2,2,3), (-9,14,26,27), (-10,18,23,27), \ldots$$

For other packings, it is false for some residue classes, and possibly true for the others.

Successive differences in missing curvaturtes for $(-4, 5, 20, 21)$

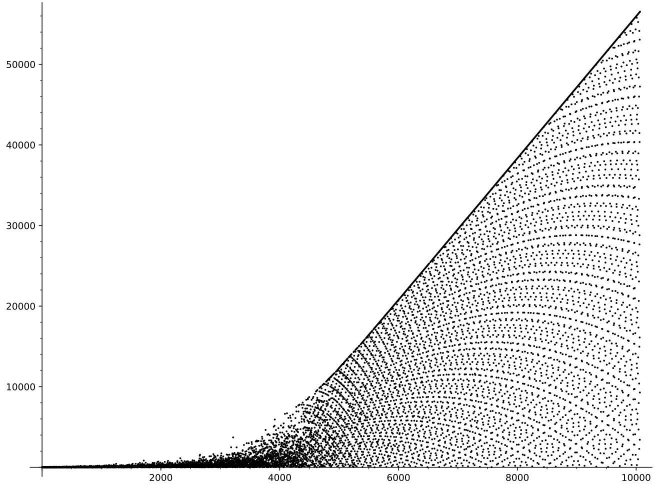


Figure 10: Type $(6, 5, 1)$

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;
- Tangent circles $\mathcal{C}_1, \mathcal{C}_2$ of coprime curvatures a, b;

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;
- Tangent circles $\mathcal{C}_1, \mathcal{C}_2$ of coprime curvatures a, b;
- $f_{\mathcal{C}}(x, y) := (a+b)x^2 + (a+b+c-d)xy + (a+c)y^2 = [A, B, C]$ of discriminant $-4a^2$.

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;
- Tangent circles $\mathcal{C}_1, \mathcal{C}_2$ of coprime curvatures a, b;
- $f_{\mathcal{C}}(x, y) := (a+b)x^2 + (a+b+c-d)xy + (a+c)y^2 = [A, B, C]$ of discriminant $-4a^2$.
- The set of curvatures of circles tangent to $\mathcal{C}_1$ are the properly represented values of $f_{\mathcal{C}} - a$;

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;
- Tangent circles $\mathcal{C}_1, \mathcal{C}_2$ of coprime curvatures a, b;
- $f_{\mathcal{C}}(x, y) := (a + b)x^2 + (a + b + c - d)xy + (a + c)y^2 = [A, B, C]$ of discriminant $-4a^2$.
- The set of curvatures of circles tangent to $\mathcal{C}_1$ are the properly represented values of $f_{\mathcal{C}} - a$;
- $f_{\mathcal{C}} - a \equiv A\left(x + \frac{B}{2A}y\right)^2 \pmod a$

Sample proof: quadratic

- For the packing $(-3, 5, 8, 8)$, all curvatures are $0, 1 \pmod 4$;
- Tangent circles $\mathcal{C}_1, \mathcal{C}_2$ of coprime curvatures a, b;
- $f_{\mathcal{C}}(x, y) := (a+b)x^2 + (a+b+c-d)xy + (a+c)y^2 = [A, B, C]$ of discriminant $-4a^2$.
- The set of curvatures of circles tangent to $\mathcal{C}_1$ are the properly represented values of $f_{\mathcal{C}} - a$;
- $f_{\mathcal{C}} - a \equiv A\left(x + \frac{B}{2A}y\right)^2 \pmod a$
- $\chi_2(\mathcal{C}_1) := \left(\frac{b}{a}\right)$ is well-defined;

Sample proof: quadratic

- Quadratic reciprocity: $\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = 1$;

Sample proof: quadratic

- Quadratic reciprocity: $\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = 1$;
- There exists paths of tangent circles with coprime curvatures between any fixed pair of circles. Thus, $\chi_2(\mathcal{A})$ is well-defined.

Sample proof: quadratic

- Quadratic reciprocity: $\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = 1$;
- There exists paths of tangent circles with coprime curvatures between any fixed pair of circles. Thus, $\chi_2(\mathcal{A})$ is well-defined.
- $\chi_2(\mathcal{A}) = \left(\frac{8}{5}\right) = -1$.

Sample proof: quadratic

- Quadratic reciprocity: $\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = 1$;
- There exists paths of tangent circles with coprime curvatures between any fixed pair of circles. Thus, $\chi_2(\mathcal{A})$ is well-defined.
- $\chi_2(\mathcal{A}) = \left(\frac{8}{5}\right) = -1$.
- Therefore squares cannot appear anywhere!

Outline of proof: quartic

- The quadratic form $f_{\mathcal{C}}(x, y)$ is the norm form of a lattice $\Lambda \subseteq \mathbb{Q}(i)$.

Outline of proof: quartic

- The quadratic form $f_{\mathcal{C}}(x, y)$ is the norm form of a lattice $\Lambda \subseteq \mathbb{Q}(i)$.
- Using an appropriate quartic reciprocity symbol on elements of Λ, we can define $\chi_4(\mathcal{C})$.

Outline of proof: quartic

- The quadratic form $f_{\mathcal{C}}(x, y)$ is the norm form of a lattice $\Lambda \subseteq \mathbb{Q}(i)$.
- Using an appropriate quartic reciprocity symbol on elements of Λ, we can define $\chi_4(\mathcal{C})$.
- In order to be well-defined and permeate through a packing, it is essential that all curvatures are either $0 \pmod 4$ or $1 \pmod 8$.

New conjecture

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, and define $S_{\mathcal{A}}$ to be the set of missing curvatures which do not lie in one of the quadratic or quartic obstruction classes. Call this set the "sporadic set" for $\mathcal{A}$. For a positive integer N, define $S_{\mathcal{A}}(N)$ to be the set of sporadic curvatures in $\mathcal{A}$ that are at most N.

New conjecture

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, and define $S_{\mathcal{A}}$ to be the set of missing curvatures which do not lie in one of the quadratic or quartic obstruction classes. Call this set the "sporadic set" for $\mathcal{A}$. For a positive integer N, define $S_{\mathcal{A}}(N)$ to be the set of sporadic curvatures in $\mathcal{A}$ that are at most N.

Conjecture

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Then $S_{\mathcal{A}}$ is finite.

New conjecture

Definition

Let $\mathcal{A}$ be a primitive Apollonian circle packing, and define $S_{\mathcal{A}}$ to be the set of missing curvatures which do not lie in one of the quadratic or quartic obstruction classes. Call this set the "sporadic set" for $\mathcal{A}$. For a positive integer N, define $S_{\mathcal{A}}(N)$ to be the set of sporadic curvatures in $\mathcal{A}$ that are at most N.

Conjecture

Let $\mathcal{A}$ be a primitive Apollonian circle packing. Then $S_{\mathcal{A}}$ is finite.

In other words, the linear, quadratic, and quartic obstructions describe all but finitely many absent curvatures.

Evidence

- I wrote efficient C code, integrated into a PARI/GP package to compute missing curvatures. This is publicly available in GitHub [Ric23a].

Evidence

- I wrote efficient C code, integrated into a PARI/GP package to compute missing curvatures. This is publicly available in GitHub [Ric23a].
- $S_{\mathcal{A}}(N)$ was computed for many packings $\mathcal{A}$ and values of N in the range $10^{10} \leq N \leq 10^{12}$.

Evidence

- I wrote efficient C code, integrated into a PARI/GP package to compute missing curvatures. This is publicly available in GitHub [Ric23a].
- $S_{\mathcal{A}}(N)$ was computed for many packings $\mathcal{A}$ and values of N in the range $10^{10} \leq N \leq 10^{12}$.
- 10^{10} typically finishes in a few hours (on a modern laptop) and requires less than 500MB of memory.

Evidence

- I wrote efficient C code, integrated into a PARI/GP package to compute missing curvatures. This is publicly available in GitHub [Ric23a].
- $S_{\mathcal{A}}(N)$ was computed for many packings $\mathcal{A}$ and values of N in the range $10^{10} \leq N \leq 10^{12}$.
- 10^{10} typically finishes in a few hours (on a modern laptop) and requires less than 500MB of memory.
- 10^{12} was done for a few packings and took just over a week and about 30GB.

Evidence

- For all 14 types that exhibit different obstruction behaviour, we took the "smallest" three packings, and computed high enough so that

$$\frac{N}{\max(S_{\mathcal{A}}(N))} \geq 10$$

for all 42 packings.

Evidence

- For all 14 types that exhibit different obstruction behaviour, we took the "smallest" three packings, and computed high enough so that

$$\frac{N}{\max(S_{\mathcal{A}}(N))} \geq 10$$

  for all 42 packings.
- This data (and more) is stored in the GitHub repository [Ric23b].

Tables

Packing	Type	N	$\lvert S_{\mathcal{A}}(N)\rvert$	$\max(S_{\mathcal{A}}(N))$	$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$
$(0, 0, 1, 1)$	$(6, 1, 1, 1)$	10^{10}	215	1199820	8334.58
$(-12, 16, 49, 49)$		10^{11}	275276	5542869468	18.04
$(-20, 36, 49, 49)$		10^{12}	2014815	55912619880	17.89
$(-8, 12, 25, 25)$	$(6, 1, 1, -1)$	10^{10}	47070	517280220	19.33
$(-12, 25, 25, 28)$		10^{11}	238268	5919707820	16.89
$(-15, 24, 40, 49)$		$2 \cdot 10^{11}$	639149	12692531688	15.75
$(-15, 28, 33, 40)$	$(6, 1, -1)$	10^{10}	80472	820523160	12.19
$(-20, 33, 52, 57)$		10^{11}	240230	4127189100	24.23
$(-23, 40, 57, 60)$		10^{11}	392800	8689511520	11.51
$(-4, 5, 20, 21)$	$(6, 5, 1)$	10^{10}	3659	32084460	311.68
$(-16, 29, 36, 45)$		10^{10}	80256	927211800	10.79
$(-19, 36, 44, 45)$		10^{11}	177902	3603790320	27.75
$(-3, 5, 8, 8)$	$(6, 5, -1)$	10^{10}	676	3122880	3202.17
$(-12, 21, 29, 32)$		10^{10}	30347	312225420	32.03
$(-19, 32, 48, 53)$		$2.5 \cdot 10^{10}$	168264	2286209460	10.94

Tables

Packing	Type	N	$\lvert S_{\mathcal{A}}(N)\rvert$	$\max(S_{\mathcal{A}}(N))$	$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$
$(-3, 4, 12, 13)$	$(6, 13, 1)$	10^{10}	731	7354464	1359.72
$(-12, 21, 28, 37)$		10^{11}	234386	3470731680	28.81
$(-11, 16, 36, 37)$		10^{10}	20748	226988340	44.06
$(-8, 13, 21, 24)$	$(6, 13, -1)$	10^{10}	5273	45348900	220.51
$(-11, 21, 24, 28)$		10^{10}	21003	176441136	56.68
$(-20, 37, 45, 52)$		10^{11}	229356	4079861484	24.51
$(-16, 32, 33, 41)$	$(6, 17, 1, 1)$	10^{10}	81777	841440840	11.88
$(-7, 8, 56, 57)$		10^{10}	55057	595231740	16.80
$(-16, 20, 81, 81)$		10^{12}	1075024	26983035480	37.06
$(-4, 8, 9, 9)$	$(6, 17, 1, -1)$	10^{10}	2057	10742460	930.89
$(-7, 9, 32, 32)$		10^{10}	34916	367956840	27.18
$(-15, 32, 32, 33)$		10^{11}	585942	8505627180	11.76
$(-7, 12, 17, 20)$	$(6, 17, -1)$	10^{10}	3744	17141220	583.39
$(-12, 17, 41, 44)$		10^{10}	31851	270186456	37.01
$(-15, 24, 41, 44)$		10^{10}	80106	803343900	12.45

Tables

Packing	Type	N	$\lvert S_{\mathcal{A}}(N)\rvert$	$\max(S_{\mathcal{A}}(N))$	$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$
$(-5, 7, 18, 18)$	$(8, 7, 1)$	10^{10}	16417	86709570	115.33
$(-6, 10, 15, 19)$		10^{10}	24305	133977255	74.64
$(-9, 18, 19, 22)$		10^{10}	14866	82815750	120.75
$(-2, 3, 6, 7)$	$(8, 7, -1)$	10^{10}	236	429039	23307.90
$(-5, 6, 30, 31)$		10^{10}	19695	97583070	102.48
$(-14, 27, 31, 34)$		$2 \cdot 10^{10}$	99294	1643827935	12.17
$(-1, 2, 2, 3)$	$(8, 11, 1)$	10^{10}	61	97287	102788.66
$(-9, 14, 26, 27)$		10^{10}	17949	85926675	116.38
$(-10, 18, 23, 27)$		10^{10}	25944	124625694	80.24
$(-6, 11, 14, 15)$	$(8, 11, -1)$	10^{10}	3381	20149335	496.29
$(-10, 14, 35, 39)$		$4 \cdot 10^{10}$	256228	2934238515	13.63
$(-13, 23, 30, 38)$		10^{10}	71341	598107510	16.72

Why was this not previously discovered?

- Lack of data: The only published paper we could find that did related computations was Fuchs-Sanden;

Why was this not previously discovered?

- Lack of data: The only published paper we could find that did related computations was Fuchs-Sanden;
- No quadratic or quartic obstructions exist for the two smallest packings, $(0, 0, 1, 1)$ and $(-1, 2, 2, 3)$;

Why was this not previously discovered?

- Lack of data: The only published paper we could find that did related computations was Fuchs-Sanden;
- No quadratic or quartic obstructions exist for the two smallest packings, $(0, 0, 1, 1)$ and $(-1, 2, 2, 3)$;
- Even if you collect data containing the obstructions, noticing the quadratic/quartic families is non-obvious as they look potentially sparse;

A dataset not analyzed from 1 year ago

m2_3_6_7 missing_4mil

File Edit View

```
Missing up to 4 million:

[18, 30, 46, 51, 78, 123, 126, 135, 162, 186, 198, 211, 219, 270, 414, 450, 510, 526, 534, 555, 591, 634, 639, 646, 651, 750, 786, 807, 819,
882, 891, 942, 963, 970, 975, 991, 1047, 1074, 1206, 1266, 1275, 1330, 1374, 1422, 1446, 1458, 1479, 1506, 1614, 1635, 1878, 1899, 1911, 1954,
1971, 2034, 2046, 2067, 2074, 2079, 2130, 2151, 2178, 2179, 2206, 2370, 2394, 2430, 2487, 2499, 2514, 2571, 2754, 2766, 2878, 2958, 3042, 3090,
3154, 3159, 3234, 3319, 3339, 3474, 3483, 3490, 3571, 3642, 3726, 4050, 4446, 4579, 4650, 4890, 5091, 5151, 5202, 5262, 5370, 5574, 5667, 5730,
5982, 5994, 6094, 6294, 6498, 6535, 6631, 6750, 6787, 7102, 7251, 7270, 7338, 7410, 7515, 7650, 7914, 7938, 8226, 8346, 8766, 8775, 8814, 9027,
9039, 9066, 9522, 9627, 9711, 9954, 10294, 10578, 11230, 11250, 11454, 11475, 11622, 11655, 11755, 11874, 12030, 12495, 12735, 12867, 13122,
13335, 13554, 13563, 13575, 14310, 14322, 14335, 14394, 14515, 14646, 15087, 15138, 15555, 15630, 15675, 16431, 16494, 16830, 17154, 17298,
17574, 18006, 18510, 18558, 18670, 19455, 19602, 20295, 20910, 21735, 21870, 22050, 22338, 22647, 22974, 23610, 24126, 24243, 24642, 24747,
25134, 26742, 27099, 27378, 27750, 27939, 29799, 30207, 30258, 30375, 30879, 30891, 30990, 31530, 31950, 33255, 33282, 34350, 35919, 36450,
36591, 38586, 39762, 40290, 40314, 42270, 43170, 43218, 43950, 45066, 45798, 45915, 46290, 46530, 46818, 50562, 51774, 52830, 53022, 53715,
54246, 54450, 54618, 55050, 55539, 58410, 58482, 58935, 59922, 62658, 64275, 66399, 66978, 71442, 71619, 76050, 78990, 80271, 80802, 83535,
85698, 86550, 89055, 90738, 94731, 95922, 100206, 101250, 101511, 102978, 106110, 106722, 109755, 111630, 112338, 116922, 118098, 118575,
122775, 124002, 127530, 130050, 136242, 142230, 142578, 149058, 154722, 155682, 156234, 162450, 163515, 169362, 176418, 183618, 190962, 198450,
205122, 206082, 209610, 213858, 214110, 221778, 229842, 238050, 246402, 247230, 254622, 254898, 263538, 272322, 281250, 290322, 299538, 308898,
318402, 328050, 337842, 347778, 357858, 368082, 378450, 388962, 399618, 410418, 421362, 429039, 432450, 443682, 455058, 466578, 478242, 490050,
502002, 514098, 526338, 538722, 551250, 563922, 576738, 589698, 602802, 616050, 629442, 642978, 656658, 670482, 684450, 698562, 712818, 727218,
741762, 756450, 771282, 786258, 801378, 816642, 832050, 847602, 863298, 879138, 895122, 911250, 927522, 943938, 960498, 977202, 994050,
1011042, 1028178, 1045458, 1062882, 1080450, 1098162, 1116018, 1134018, 1152162, 1170450, 1188882, 1207458, 1226178, 1245042, 1264050, 1283202,
1302498, 1321938, 1341522, 1361250, 1381122, 1401138, 1421298, 1441602, 1462050, 1482642, 1503378, 1524258, 1545282, 1566450, 1587762, 1609218,
1630818, 1652562, 1674450, 1696482, 1718658, 1740978, 1763442, 1786050, 1808802, 1831698, 1854738, 1877922, 1901250, 1924722, 1948338, 1972098,
1996002, 2020050, 2044242, 2068578, 2093058, 2117682, 2142450, 2167362, 2192418, 2217618, 2242962, 2268450, 2294082, 2319858, 2345778, 2371842,
2398050, 2424402, 2450898, 2477538, 2504322, 2531250, 2558322, 2585538, 2612898, 2640402, 2668050, 2695842, 2723778, 2751858, 2780082, 2808450,
2836962, 2865618, 2894418, 2923362, 2952450, 2981682, 3011058, 3040578, 3070242, 3100050, 3130002, 3160098, 3190338, 3220722, 3251250, 3281922,
3312738, 3343698, 3374802, 3406050, 3437442, 3468978, 3500658, 3532482, 3564450, 3596562, 3628818, 3661218, 3693762, 3726450, 3759282, 3792258,
3825378, 3858642, 3892050, 3925602, 3959298, 3993138]

Residue counts: [[3, 6, 7, 10, 15, 18, 19, 22], Vecsmall([41, 67, 5, 7, 43, 291, 7, 11])]
The last 160 residues are all 18.
```

Ln 1, Col 1 | 100% | Windows (CRLF) | UTF-8

The future

- The local-global conjecture is still open for many packings and residue classes;

The future

- The local-global conjecture is still open for many packings and residue classes;
- It is also still open for primes, squarefree numbers, etc;

The future

- The local-global conjecture is still open for many packings and residue classes;
- It is also still open for primes, squarefree numbers, etc;
- Variants of the reciprocity obstructions exist in other thin (semi)group orbits.

The future

- The local-global conjecture is still open for many packings and residue classes;
- It is also still open for primes, squarefree numbers, etc;
- Variants of the reciprocity obstructions exist in other thin (semi)group orbits.
- What is the most general form of a "reciprocity obstruction"? Can it only forbid power sets from appearing?

The future

- The local-global conjecture is still open for many packings and residue classes;
- It is also still open for primes, squarefree numbers, etc;
- Variants of the reciprocity obstructions exist in other thin (semi)group orbits.
- What is the most general form of a "reciprocity obstruction"? Can it only forbid power sets from appearing?
- Is there a connection to the Brauer-Manin obstruction?

Thin (semi)groups

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be a thin (semi)group, let $v \in \mathbb{Z}^n$ be a primitive vector, and let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional;

Thin (semi)groups

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be a thin (semi)group, let $v \in \mathbb{Z}^n$ be a primitive vector, and let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional;
- The values of the orbit $L(\Gamma \cdot v)$ may have congruence obstructions modulo some d;

Thin (semi)groups

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be a thin (semi)group, let $v \in \mathbb{Z}^n$ be a primitive vector, and let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional;
- The values of the orbit $L(\Gamma \cdot v)$ may have congruence obstructions modulo some d;
- The orbit is fractal-like, and has a corresponding Hausdorff dimension δ;

Thin (semi)groups

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be a thin (semi)group, let $v \in \mathbb{Z}^n$ be a primitive vector, and let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional;
- The values of the orbit $L(\Gamma \cdot v)$ may have congruence obstructions modulo some d;
- The orbit is fractal-like, and has a corresponding Hausdorff dimension δ;
- Previously, it was believed that if δ was large enough (typically $\delta > 1/2$, so the average multiplicity of a value tends to ∞), then eventually all numbers obeying the congruence obstruction would appear;

Thin (semi)groups

- Let $\Gamma \subseteq \mathrm{GL}(n, \mathbb{Z})$ be a thin (semi)group, let $v \in \mathbb{Z}^n$ be a primitive vector, and let $L : \mathbb{Z}^n \to \mathbb{Z}$ be a linear functional;
- The values of the orbit $L(\Gamma \cdot v)$ may have congruence obstructions modulo some d;
- The orbit is fractal-like, and has a corresponding Hausdorff dimension δ;
- Previously, it was believed that if δ was large enough (typically $\delta > 1/2$, so the average multiplicity of a value tends to ∞), then eventually all numbers obeying the congruence obstruction would appear;
- Our work disproves this for the Apollonian circle packing case, and we have further examples in the context of Zaremba's conjecture and thin semigroups of $\mathrm{GL}(2, \mathbb{Z})$.

Acknowledgments and References

This research was partially supported by NSF-CAREER CNS-1652238 (PI Katherine E. Stange).
All computations were done with PARI/GP ([PAR23]).

Jean Bourgain and Alex Kontorovich.
On the local-global conjecture for integral Apollonian gaskets.
Invent. Math., 196(3):589–650, 2014.
With an appendix by Péter P. Varjú.

Elena Fuchs and Katherine Sanden.
Some experiments with integral Apollonian circle packings.
Exp. Math., 20(4):380–399, 2011.

Elena Fuchs.
Arithmetic properties of Apollonian circle packings.
ProQuest LLC, Ann Arbor, MI, 2010.
Thesis (Ph.D.)–Princeton University.

Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan.
Apollonian Circle Packings: Number Theory.
J. Number Theory, 100(1):1–45, 2003.

The PARI Group, Univ. Bordeaux.
PARI/GP version 2.16.1, 2023.
available from https://pari.math.u-bordeaux.fr/.

James Rickards.
Apollonian.
https://github.com/JamesRickards-Canada/Apollonian, 2023.

James Rickards.
Apollonian Missing Curvatures.
https://github.com/JamesRickards-Canada/Apollonian-Missing-Curvatures, 2023.