Totally geodesic surfaces in Bianchi orbifolds

James Rickards

Saint Mary's University

james.rickards@smu.ca

13 December 2024

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- Arithmetic Fuchsian groups: arise from quaternion algebras

Quaternion algebras

- F a field, $\mathrm{char}(F) \neq 2$, $a, b \in F^\times$
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- Can be realized as a subalgebra of $\mathrm{Mat}(2, F(\sqrt{a}))$:

$$i \to \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad j \to \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

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- Gives an embedding

$$\sigma : B \to \mathrm{Mat}(2, \mathbb{R}),$$

unique up to conjugation.

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- Except for $\mathrm{PSL}(2, \mathbb{Z})$, these are always co-compact

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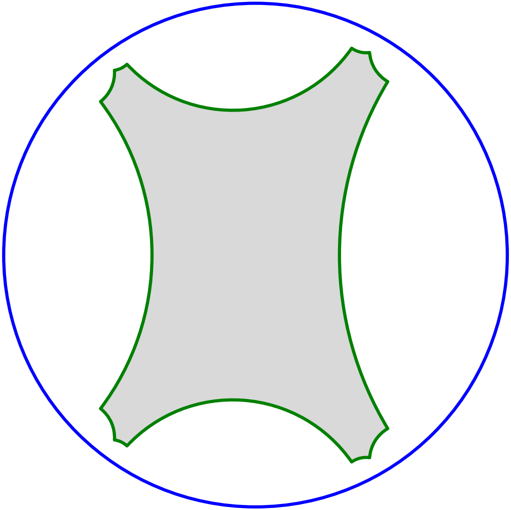
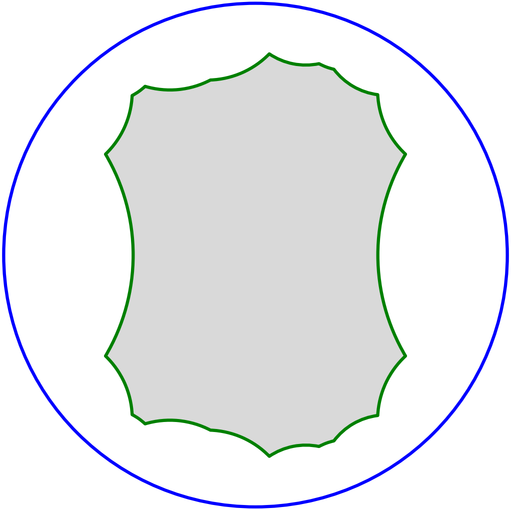
- In every orbit Γz, pick the point closest to p.
- Connected region whose boundary is a closed hyperbolic polygon with finitely many sides, which come paired.

Example 1



$F = \mathbb{Q}$, $\mathfrak{D} = 21$.

Example 2



$F = \mathbb{Q}(\sqrt{5})$, $\mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{D}) = 61$.

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- Page (2015) higher dimensional variant (Kleinian groups), much improved algebra step, also implemented in Magma
- R. (2022) improved geometric implementation, used Page's algebraic improvements, implemented in PARI/GP

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- Naïve algorithm: $O(n^2)$.
- Graham's scan: $O(n \log(n))$

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- Example:

$$B = \left(\frac{-1, 33}{\mathbb{Q}}\right), \quad \mathcal{O} = \left\langle 1, i, \frac{1+j}{2}, \frac{1+i+j+k}{2} \right\rangle_{\mathbb{Z}}$$

- Find $e, f, g, h \in \mathbb{Z}$ satisfying

$$e^2 + f^2 - 8g^2 - 16h^2 + eg + eh + fh - 16gh = 1$$

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- Probabilistic method influenced by geometry

Sample timings

deg(F)	Genus	Area	$\approx$t(MAGMA)	t(PARI) (s)	$\approx \frac{\text{t(MAGMA)}}{\text{t(PARI)}}$
1	9	100.530	80 s	0.028	2900
1	81	1005.310	$>$ 16 days	0.388	$>$3500000
1	161	2023.186	$>$ 20 days	0.976	$>$1700000
1	403	5051.681	$>$ 30 days	3.095	$>$830000
2	41	542.867	5 h	0.501	36000
2	163	2067.168	$>$ 16 days	5.230	$>$260000
3	27	350.462	1.3 h	0.861	5700
3	61	770.737	2.4 h	2.486	3500
4	37	469.145	2.2 h	2.852	2700
4	129	1633.628	7.4 days	24.637	26000
5	27	343.481	4.9 h	3.499	5000
5	67	829.380	43 h	12.873	12000
6	14	198.968	1.1 h	2.657	1500
6	42	542.448	6.4 h	14.103	1600
7	6	138.230	7.6 m	3.892	120
7	13	238.761	1.5 h	7.150	760
8	29	422.230	1.1 h	34.929	120
8	27	423.068	1.3 h	43.681	110

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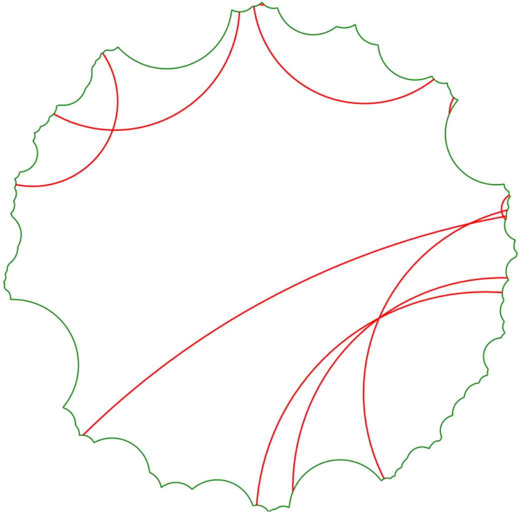
- All closed geodesics arise this way
- Rational arithmetic Fuchsian groups - arise from optimal embeddings:

$$\phi : \mathcal{O}_D := \mathbb{Z} + \frac{p_D + \sqrt{D}}{2}\mathbb{Z} \to \mathrm{O}$$

Example



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- Inversions in a circle $\to$ inversions in orthogonal sphere

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- Example: $\mathrm{PSL}(2, \mathcal{O}_D)$ with $D < 0$ (Bianchi groups)

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- From now on, we focus on $\Gamma_D := \mathrm{PSL}(2, \mathcal{O}_D)$ for $D < 0$ a fundamental discriminant

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  - **Embedded:** No self-intersection

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- Closed: iff $\left(\frac{D,\Delta}{\mathbb{Q}}\right) \not\cong \mathrm{Mat}(2, \mathbb{Q})$

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- This list is complete if $\mathrm{Cl}(\mathcal{O}_D)$ has no non-trivial 4-torsion.

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- Applied this result to Vulakh's parametrizations

Embedded closed TGS II

- Let $D' = \begin{cases} D/4 & \text{if } D \text{ is even} \\ D & \text{if } D \text{ is odd} \end{cases}$
- For each integer m satisfying
  - $1 \leq m < D'/2$
  - $1 \leq m^2 \pmod{D'} < D'/4$
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- The number of ECTGS is $>> D$ and $<< D^{1+\epsilon}$

Conjectural list of D with no ECTGS (Jung-Reid)

No 4-torsion in class group:

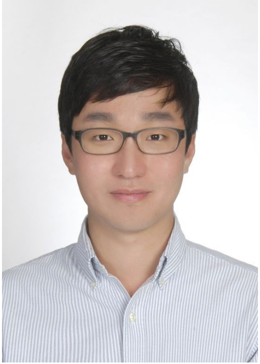
$$\{-3, -4, -7, -8, -11, -15, -19, -20, -23, -24, -31, -35, -40, -47, -51, \\ -59, -71, -79, -84, -87, -104, -119, -131, -143, -159, -167, -191, -215, \\ -231, -239, -287, -311, -359, -479, -551, -671, -719\}^1$$

4-torsion in class group:

$$\{-39, -55, -56, -68, -95, -111, -136, -164\}^1$$

[1] Corrected published lists

Collaborators



(a) Junehyuk Jung

(b) Sam Kim

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  - Compute signed intersection number with all geodesics to determine if separating
- Last 3 steps all completed in PARI/GP

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- Not a single one was separating

Number of ECTGS

The number of ECTGS grows like D, but can we say more?

D	$h(D)$	#ECTGS
-1956	20	28
-1959	42	24
-1963	6	151
-1967	36	38
-1972	12	31
-1976	28	19
-1979	23	51
-1983	16	118
-1987	7	141
-1988	24	21
-1991	56	6

Number of ECTGS: Class number maxima

D	$h(D)$	#ECTGS	Avg # ECTGS
-3	1	0	0
-15	2	0	0
-23	3	0	0
-39	4	0	0
-47	5	0	0.125
-71	7	0	0.292
-95	8	0	0.419
-119	10	0	0.605
-167	11	0	1.074
-191	13	0	1.283
-215	14	0	1.530
-239	15	0	1.919
-311	19	0	2.694
-431	21	1	4.189

Number of ECTGS: Class number maxima

D	$h(D)$	#ECTGS	Avg # ECTGS
-479	25	0	4.723
-551	26	0	5.744
-671	30	0	7.597
-719	31	0	8.357
-791	32	2	9.444
-839	33	2	10.239
-959	36	3	11.986
-1151	41	2	14.915
-1319	45	2	17.871
-1511	49	2	21.046
-1559	51	1	21.949
-1679	52	4	23.908
-1991	56	6	29.370

Number of ECTGS: #ECTGS maxima

D	$h(D)$	#ECTGS	Avg $h(D)$
-3	1	0	1
-43	1	2	1.8
-67	1	4	2.182
-115	2	6	3.139
-163	1	14	3.769
-235	2	18	4.479
-379	3	24	5.786
-403	2	31	6
-427	2	33	6.159
-499	3	35	6.701
-547	3	39	7.054
-643	3	50	7.617
-795	4	57	8.553
-883	3	74	9.018
-907	3	75	9.168

Number of ECTGS: #ECTGS maxima

D	$h(D)$	#ECTGS	Avg $h(D)$
-955	4	77	9.439
-1027	4	80	9.732
-1227	4	96	10.641
-1387	4	110	11.334
-1411	4	118	11.492
-1435	4	120	11.510
-1507	4	121	11.852
-1555	4	139	12.047
-1867	5	154	13.221

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- Example: $D = -43$, 2 ECTGS, but they are linearly dependent
- Example: $D = -451$, 11 ECTGS, linearly independent
- Observation: if D is even, then all ECTGS found are linearly independent

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- Problem: requires better computational geometry algorithms

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- Similar geometric improvements are relevant to computing the fundamental domain, e.g. 3D convex hull algorithms

ECTGS