Matthew Brennan's Published Problems

1 CMO

The Canadian Mathematical Olympiad is Canada's premier olympiad style contest, and is typically written in March. Approximately 80-100 students qualify for it, and the format is 5 problems in 3 hours.

1. (2013 CMO 3) Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90°$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

2. (2013 CMO 5) Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

3. (2014 CMO 3) Let p be a fixed odd prime. A $p-$tuple $(a_1, a_2, \ldots, a_p)$ of integers is said to be *good* if

   (i) $0 \leq a_i \leq p - 1$ for all i, and

   (ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by p, and

   (iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots a_pa_1$ is divisible by p.

   Determine the number of good $p-$tuples.

4. (2014 CMO 4) The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.

5. (2015 CMO 1) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n - 1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.

6. (2015 CMO 3) On a $(4n+2)\times(4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

7. (2015 CMO 4) Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.

8. (2016 CMO 3) Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n)$ is a prime number for infinitely many integers n.

9. (2017 CMO 2) Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6)) = 4$ and $f(f(25)) = 3$. Prove that if p is prime then $f(p)$ is also prime.

10. (2017 CMO 3) Let n be a positive integer, and define $S_n = 1, 2, \ldots, n$. Consider a non-empty subset T of S_n. We say that T is balanced if the median of T is equal to the average of T. For example, for $n = 9$, each of the subsets $\{7\}$, $\{2, 5\}$, $\{2, 3, 4\}$, $\{5, 6, 8, 9\}$, and $\{1, 4, 5, 7, 8\}$ is balanced; however, the subsets $\{2, 4, 5\}$ and $\{1, 2, 3, 5\}$ are not balanced. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.

11. (2017 CMO 4) Points P and Q lie inside parallelogram $ABCD$ and are such that triangles ABP and BCQ are equilateral. Prove that the line through P perpendicular to DP and the line through Q perpendicular to DQ meet on the altitude from B in triangle ABC.

12. (2017 CMO 5) One hundred circles of radius one are positioned in the plane so that the area of any triangle formed by the centres of three of these circles is at most 2017. Prove that there is a line intersecting at least three of these circles.

13. (2018 CMO 5) Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choice Sarah makes, her number N will at some point be divisible by 2018.

14. (2019 CMO 1, with David Arthur) Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

    (a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

    (b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

    Remark 1. This was originally proposed as a much harder problem: Let S be a set of at least three points in the plane satisfying that for any three distinct points in S that are not collinear, the center of the circle passing through these three points is also in S. Does S necessarily contain a point within distance one of the origin?

15. (2019 CMO 2) Let a and b be positive integers such that $a+b^3$ is divisible by $a^2+3ab+3b^2-1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

16. (2019 CMO 4) Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

17. (2019 CMO 5) David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

18. (2020 CMO 1) Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n - 2$.

19. (2020 CMO 3) A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

20. (2021 CMO 2) Let $n \geq 2$ be a positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying that $a_1 + a_2 + \cdots + a_n = 2^n - 1$. Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \cdots + \frac{a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}}$$

21. (2021 CMO 3, with James Rickards) At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. Two guests will chat if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will talk to one another.

22. (2021 CMO 4, with James Rickards) A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

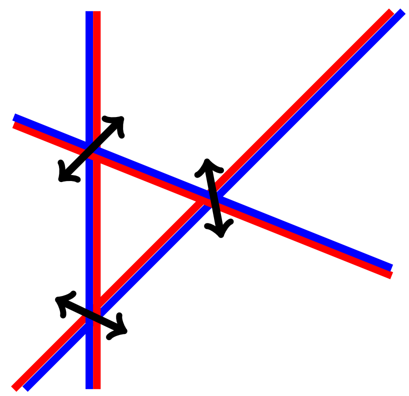
$$\gcd(f(f(x)), f(x + y)) = \gcd(x, y)$$

for all pairs of positive integers x, y. Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

2 IMOSL

The International Mathematical Olympiad is an annual competition, where countries send teams of up to 6 of their top students in high school and below. The test is two days, with each day being 3 questions in 4.5 hours. The test is chosen by the jury from a shortlist prepared by the problem selection committee. This shortlist is typically divided into about 8 problems in each of 4 categories (algebra, combinatorics, geometry, and number theory).

1. (2019 C4, with James Rickards) A labyrinth in Camelot consists of n walls on top of a plane, each of which is a fixed line, no two of which are parallel and no three of which have a common point. Merlin first paints one side of each wall entirely red and the other entirely blue. At the intersection of two walls, there is a two-way door connecting the two corners at which sides of different colours meet. For example, if a two-way arrow denotes a door then a possible labyrinth with $n = 3$ walls is as follows.



   After Merlin paints the walls, Morgana places k knights in the labyrinth. The knights cannot walk across walls but can walk through doors. What is the largest number k such that no matter how Merlin paints the labyrinth, Morgana can place the k knights so that it is not possible for any two of them to meet?

2. (2019 N7) Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn \log n$ pairwise coprime nth powers.

3 Other

1. (2011 Canadian Students Math Olympiad P4, on AOPS) Circles Γ_1 and Γ_2 have centers O_1 and O_2 and intersect at P and Q. A line through P intersects Γ_1 and Γ_2 at A and B, respectively, such that AB is not perpendicular to PQ. Let X be the point on PQ such that $XA = XB$ and let Y be the point within AO_1O_2B such that AYO_1 and BYO_2 are similar. Prove that $2\angle O_1AY = \angle AXB$.

2. (2018 COMC C4, with James Rickards) Given a positive integer N, Matt writes N in decimal on a blackboard, without writing any of the leading 0s. Every minute he is allowed to take two consecutive digits, erase them, and replace them with the last digit of their product. Any leading zeroes created this way are also erased. He can repeat this process for as long he likes. We call the positive integer M *obtainable* from N if starting from N, there is a finite sequence of moves that Matt can make to produce the number M. For example, 10 is obtainable from 251023 via

$$2510\underline{23} \to \underline{25}106 \to 1\underline{06} \to 10$$

   (a) Show that 2018 is obtainable from 2567777899.

(b) Find two positive integers A and B for which there is no positive integer C such both A and B are obtainable from C.

(c) Let S be any finite set of positive integers, none of which contains the digit 5 in its decimal representation. Prove that there exists a positive integer N for which all elements of S are obtainable from N.