James Rickards' Published Problems

James Rickards

1 COMC(R)

The Canadian Mathematics Open Challenge (COMC) is the main qualification test to the Canadian Mathematical Olympiad, and is typically written in November. The top 50 students qualify directly, and another 100 qualify for the Repechage (COMCR), from which another 30 students are chosen (qualification numbers are all approximate). There are 3 sections (A, B, C) of 4 problems each. The problems in part A are worth 4 points, in part B are worth 6 points, and in part C are worth 10 points. There are 2.5 hours to complete the contest, and the typical CMO qualification score is somewhere in the range 60-70 points (out of 80).

1.1 COMC

1. (2013 C2c) Find all triples of real numbers (a, b, c) such that

$$a^2 + b^2 + c = b^2 + c^2 + a = c^2 + a^2 + b.$$

2. (2015 C4) Mr Whitlock decides to play a game with his math class. He divides c dollars among his class of n students ($c \geq 1$, $n \geq 2$), where each student gets a non-negative integral amount of money (the money is not necessarily divided equally). At any step, let R be the set of richest people and P be the set of poorest people. Then each person in R gives one dollar to each person in P (people are allowed to go in debt, i.e. negative money). The game ends when everyone has the same amount of money, or we reach a distribution of money that previously occured.

   (a) Say $n = 2$ and the students are David and Jacob, where David is at least as rich as Jacob to begin. Show that the game ends and find the distribution of money at the end.

   (b) Give an example of a game that ends with at least one student in debt.

   (c) In all cases, prove that the game ends.

3. (2017 C4c) Let n be a positive integer, and $S_n = \{1, 2, \ldots, 2n\}$. A *perfect pairing* of S_n is defined to be a partitioning of the $2n$ numbers into n pairs, such that the sum of the two numbers in each pair is a perfect square. Prove or disprove: there exists a positive integer n for which S_n has at least 2017 different perfect pairings.

4. (2018 B4) Let n be a positive integer, and S the set of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that:

   (a) $x_i \geq i$ for $1 \leq i \leq n$;

(b) $\sum_{i=1}^{n} x_i = n^2$.

Find the size of S.

5. (2018 C4, with Matt Brennan) Given a positive integer N, Matt writes N in decimal on a blackboard, without writing any of the leading 0s. Every minute he is allowed to take two consecutive digits, erase them, and replace them with the last digit of their product. Any leading zeroes created this way are also erased. He can repeat this process for as long he likes. We call the positive integer M *obtainable* from N if starting from N, there is a finite sequence of moves that Matt can make to produce the number M. For example, 10 is obtainable from 251023 via

$$2510\underline{23} \to \underline{25}106 \to 1\underline{06} \to 10$$

   (a) Show that 2018 is obtainable from 2567777899.

   (b) Find two positive integers A and B for which there is no positive integer C such both A and B are obtainable from C.

   (c) Let S be any finite set of positive integers, none of which contains the digit 5 in its decimal representation. Prove that there exists a positive integer N for which all elements of S are obtainable from N.

6. (2019 C3) Let N be a positive integer. A "good division" is a partition of $\{1, 2, \dots, N\}$ into two disjoint non-empty subsets S_1, S_2 such that the sum of the numbers in S_1 equals the product of the numbers in S_2. For example, if $N = 5$, then

$$S_1 = \{3, 5\}, \qquad S_2 = \{1, 2, 4\}$$

would be a good division.

   (i) Find a good division of $N = 7$.

   (ii) Find an N which admits two distinct good divisions.

   (iii) Show that if $N \geq 5$, then a good division exists.

7. (2020 C4) Let $S = \{4, 8, 9, 16, \dots\}$ be the set of integers of the form m^k for integers $m, k \geq 2$. For a positive integer n, let $f(n)$ denote the number of ways to write n as the sum of *distinct* elements of S. For example, $f(5) = 0$ since there are no ways to express 5 in this fashion, and $f(17) = 1$ since $17 = 8 + 9$ is the only way to express 17.

   (i) Prove that $f(30) = 0$.

   (ii) Show that $f(n) \geq 1$ for $n \geq 31$.

   (iii) Let T be the set of integers for which $f(n) = 3$. Prove that T is finite and non-empty, and find the largest element of T.

1.2 COMCR

1. (2014 P1) Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two variable polynomial in x and y, prove that it must be constant.
2. (2014 P5) Let $f(x) = x^4 + 2x^3 - x - 1$.
   (a) Prove that $f(x)$ is irreducible (an irreducible polynomial cannot be factored as the product of two non-constant polynomials with integer coefficients).
   (b) Find the exact values of the 4 roots of $f(x)$.
3. (2016 P1)
   (a) Find all positive integers n such that $11 \mid 3^n + 4^n$.
   (b) Find all positive integers n such that $31 \mid 4^n + 7^n + 20^n$.
4. (2018 P7) Let n be a positive integer, with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ with $p_1 \ldots, p_r$ distinct primes, and $e_i \geq 1$ positive integers. Define $\mathrm{rad}(n) = p_1 p_2 \cdots p_r$, the product of all distinct prime factors of n. Find all polynomials $P(x)$ with rational coefficients such that there exist infinitely many positive integers n with $P(n) = \mathrm{rad}(n)$.
5. (2019 P3) Let $f(x) = x^3 + 3x^2 - 1$ have roots a, b, c.
   (a) Find the value of $a^3 + b^3 + c^3$.
   (b) Find all possible values of $a^2b + b^2c + c^2a$.
6. (2019 P5) Let (m, n, N) be a triple of positive integers. Bruce and Duncan play a game on an $m \times n$ array, where the entries are initially all zeroes. The game follows the following rules:
   (a) They alternate turns, with Bruce going first.
   (b) On Bruce's turn, he picks a row and either adds or subtracts 1 from all of the entries in the row.
   (c) On Duncan's turn, he picks a column, and either adds or subtracts 1 from all of the entries in the column.
   (d) Bruce wins if at some point there is an entry x of the array with $|x| \geq N$.

   Find all triples (m, n, N) such that no matter how Duncan plays, Bruce has a winning strategy.

2 CMO

The Canadian Mathematical Olympiad is Canada's premier olympiad style contest, and is typically written in March. Approximately 80-100 students qualify for it, and the format is 5 problems in 3 hours.

1. (2020 CMO 4) Let $S = \{1, 4, 8, 9, 16, \ldots\}$ be the set of perfect powers of integers, i.e. numbers of the form n^k where n, k are positive integers and $k \geq 2$. Write $S = \{a_1, a_2, a_3, \ldots\}$ with terms in increasing order, so that $a_1 < a_2 < a_3 < \cdots$. Prove that there exist infinitely many integers m such that 9999 divides the difference $a_{m+1} - a_m$.

2. (2021 CMO 3, with Matt Brennan) At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. Two guests will chat if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will talk to one another.

3. (2021 CMO 4, with Matt Brennan) A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd\left(x, y\right)$$

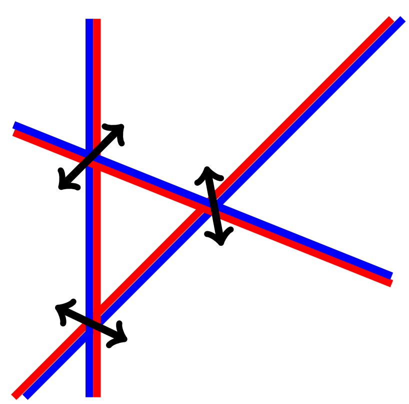
for all pairs of positive integers x, y. Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

3 IMOSL

The International Mathematical Olympiad is an annual competition, where countries send teams of up to 6 of their top students in high school and below. The test is two days, with each day being 3 questions in 4.5 hours. The test is chosen by the jury from a shortlist prepared by the problem selection committee. This shortlist is typically divided into about 8 problems in each of 4 categories (algebra, combinatorics, geometry, and number theory).

1. (2015 N7) Let $f : \mathbb{N} \to \mathbb{N}$ be any function from the natural numbers to the natural numbers. For any positive integers m, n, define $G(m,n) = \gcd(f(m)+n, f(n)+m)$. Show that $k = 2$ is the smallest positive integer k such that there exists a function f such that $G(m,n) \leq k$ for all $m \neq n$.

2. (2019 C4, with Matt Brennan) A labyrinth in Camelot consists of n walls on top of a plane, each of which is a fixed line, no two of which are parallel and no three of which have a common point. Merlin first paints one side of each wall entirely red and the other entirely blue. At the intersection of two walls, there is a two-way door connecting the two corners at which sides of different colours meet. For example, if a two-way arrow denotes a door then a possible labyrinth with $n = 3$ walls is as follows.



After Merlin paints the walls, Morgana places k knights in the labyrinth. The knights cannot walk across walls but can walk through doors. What is the largest number k such that no matter how Merlin paints the labyrinth, Morgana can place the k knights so that it is not possible for any two of them to meet?

4 Other

1. (2011 Canadian Students Math Olympiad P2, on AOPS) For a fixed positive integer k, prove that there exist infinitely many primes p such that there is an integer w, where $w^2 - 1$ is not divisible by p, and the order of w in modulus p is the same as the order of w in modulus p^k.

2. (2015 Winter Camp Buffet Contest) Let N be any positive integer. Prove there exists a positive integer x such that x^2 in base 10 starts with the digits of N.