# Eisenstein Circle Packings

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joint work with

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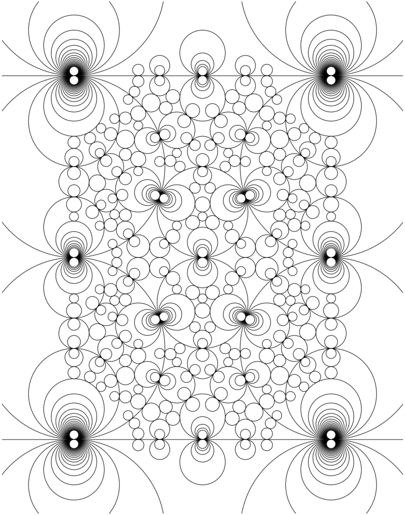
# Background: $K-$Schmidt arrangements

- $K$ is an imaginary quadratic field
- $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
- Elements of $\mathrm{PSL}(2, \mathbb{C})$ act on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as Möbius maps
- These maps preserve circles

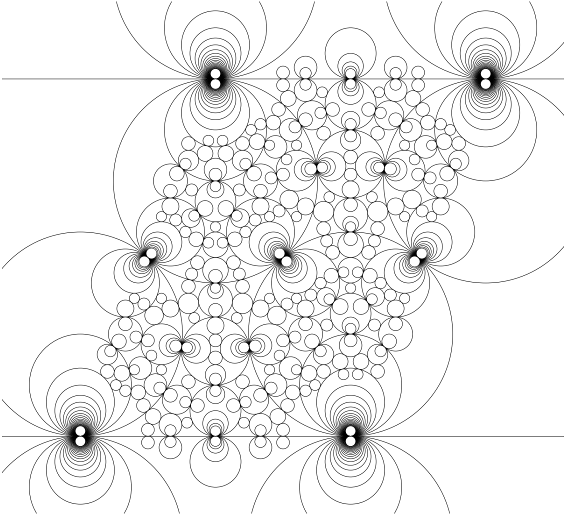
**Definition**

The $K-$Schmidt arrangement is the collection of circles $\mathrm{PSL}(2, \mathcal{O}_K)(\widehat{\mathbb{R}})$.

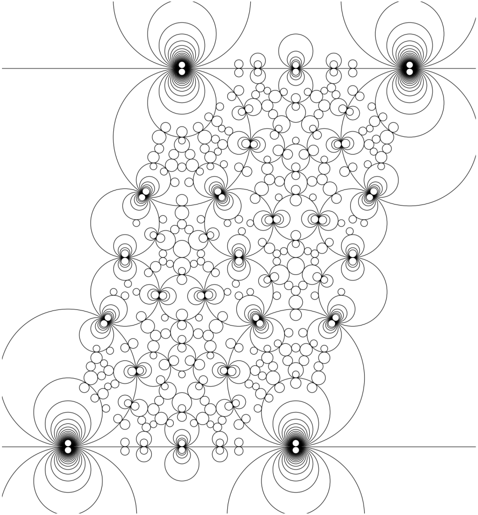
# $\mathbb{Q}(\sqrt{-2})$



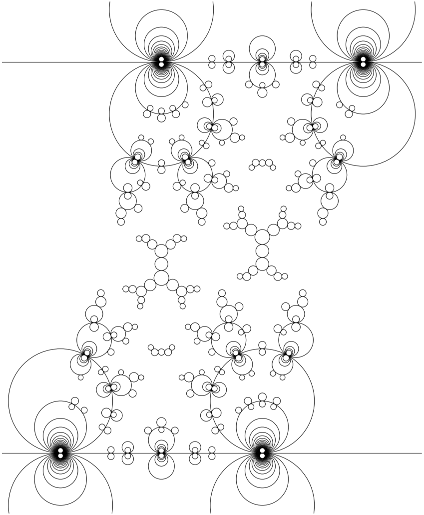
# $\mathbb{Q}(\sqrt{-7})$



# $\mathbb{Q}(\sqrt{-11})$



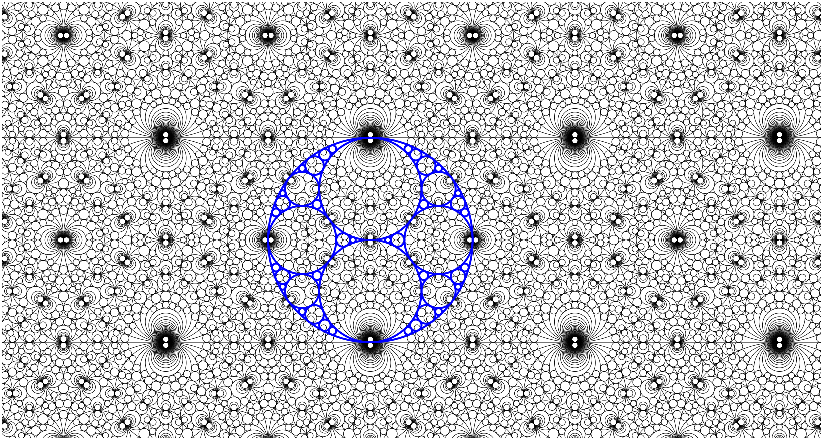
# $\mathbb{Q}(\sqrt{-15})$



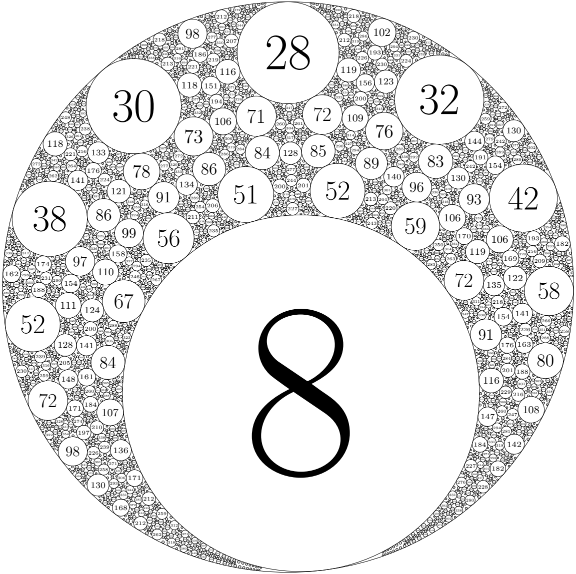
# Schmidt arrangement facts

- Usually* circles intersect at most tangentially
- Picture is connected iff $\mathcal{O}_K$ is Euclidean
- Can scale picture to make all curvatures integral
- By greedily taking immediately tangent circles, can form $K$-Apollonian circle packings as immediate tangency subgraphs

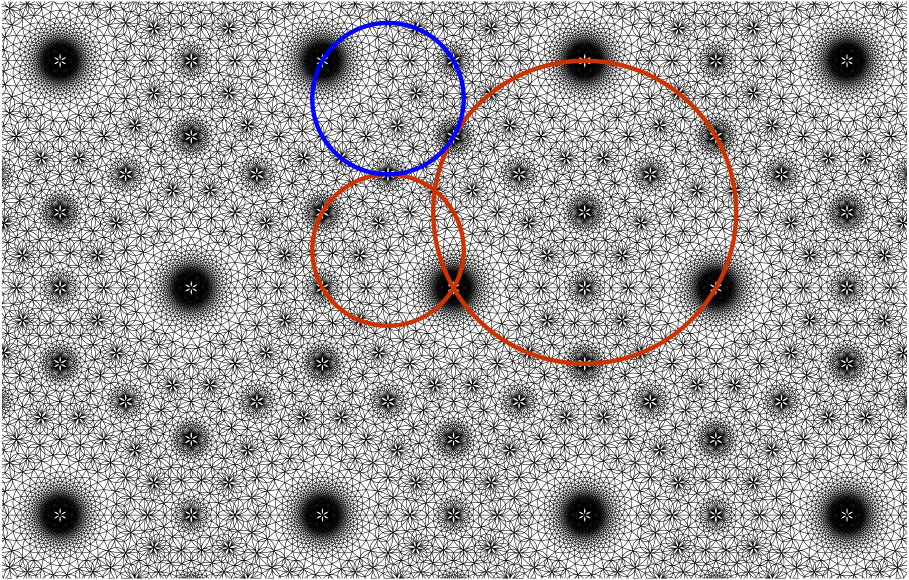
# Apollonian example, $\mathbb{Q}(i)$



# Immediate tangency packing, $\mathbb{Q}(\sqrt{-11})$



# Schmidt arrangement, $\mathbb{Q}(\sqrt{-3})$



# Issues with $\mathbb{Q}(\sqrt{-3})$

- Circles can intersect at $60^\circ$ and $120^\circ$ angles
- Immediate tangency graph includes intersections
- Root cause: non-trivial units!
- Let $\omega = \frac{1+\sqrt{-3}}{2}$. Then,

$$\omega^6 = 1, \qquad \mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[\omega].$$

- The matrix

$$\begin{pmatrix} \omega & 0 \\ 0 & 1-\omega \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}[\omega])$$

  induces a rotation by $120^\circ$

# Goals of the project

- Find a suitable adjustment of the theory to remove non-tangential intersections
- Extract the immediate tangency packings (called *Eisenstein circle packings*)
- Setup the basic theory of Eisenstein circle packings
- Implement a computational package to allow exploration of these arrangements/packings (in PARI/GP)
- Focus on searching for *reciprocity obstructions* in the circle curvatures
- Hope to find cubic reciprocity!

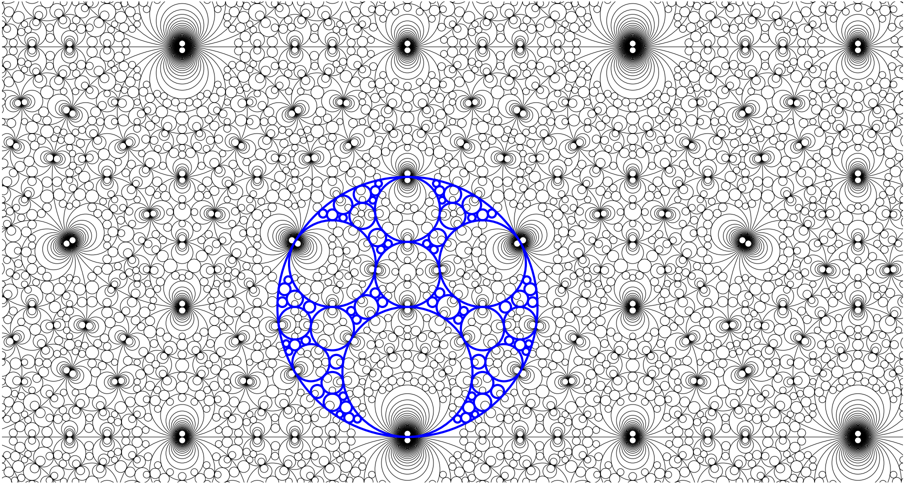
# The Eisenpint Schmidt arrangement

**Definition**

$$\Gamma_E := \left\{ M \in \mathrm{PSL}(2, \mathbb{Z}[\omega]) : M \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \pmod 2 \right\}.$$

- Index 15 in $\mathrm{PSL}(2, \mathbb{Z}[\omega])$
- Every circle tangency in the full Schmidt arrangement can be described by $\Gamma_E(\widehat{\mathbb{R}})$
- Call the resulting picture the Eisenpint Schmidt arrangement

# The Eisenpint Schmidt arrangement



# Inversive distance

**Definition**

The *inversive distance* between two circles of curvature 1 whose centres are distance $d$ apart is
$$1 - \frac{d^2}{2}.$$

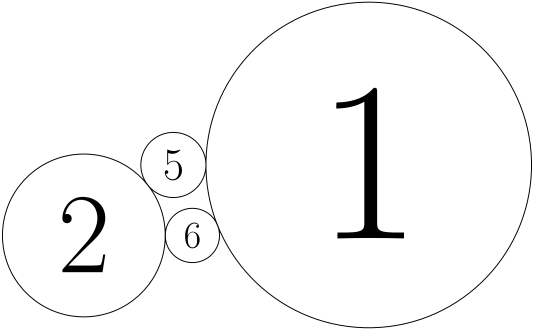
- General formula is
$$\frac{1}{2}\left(\frac{u_1}{u_2} + \frac{u_2}{u_1} - d^2 u_1 u_2\right).$$
- Preserved by Möbius maps
- $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle = -1$ iff the circles are externally tangent

# 4-wheels

**Definition**

A 4-wheel is a tuple of circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ where each circle is tangent to the next (including $\mathcal{C}_4$ to $\mathcal{C}_1$), and

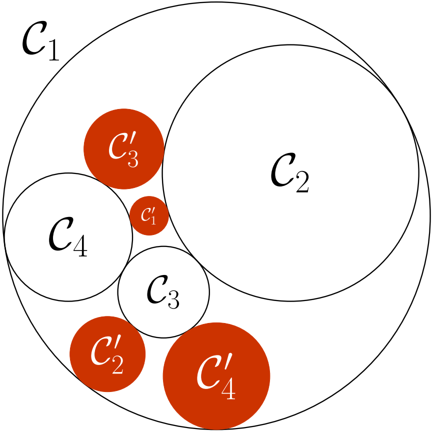
$$\langle \mathcal{C}_1, \mathcal{C}_3 \rangle = -2, \quad \langle \mathcal{C}_2, \mathcal{C}_4 \rangle = -2$$



# Circle swaps

## Definition

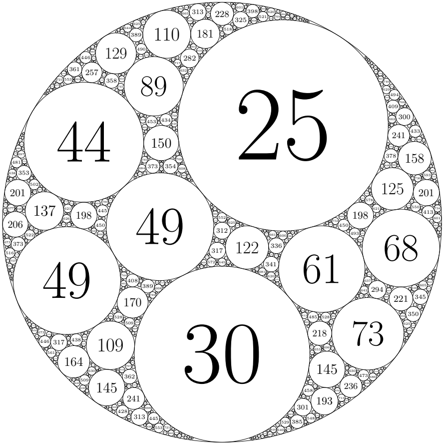
$S_4$: Inversion in the circle orthogonal to $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ swaps $\mathcal{C}_4$ to $\mathcal{C}_4'$



# Eisenstein circle packings

## Definition

An Eisenstein circle packing is created by taking a 4-wheel, applying all swaps, and iterating ad infinitum.



# Curvatures, I

**Definition**

An *Eisenstein quadruple* is the tuple of curvatures in a 4-wheel

**Proposition**

$(a, b, c, d)$ is an Eisenstein quadruple if and only if

$$a^2 + b^2 + c^2 + d^2 = 2(a + c)(b + d), \quad a + b + c + d > 0.$$

**Proposition**

The swap $S_4$ acts as

$$(a, b, c, d) \to (a, b, c, 2a + 2c - d).$$

# Curvatures, II

- $S_1$ and $S_3$ commute, as well as $S_2$ and $S_4$
- If $a, b, c, d \in \mathbb{Z}$, all other curvatures in the corresponding Eisenstein circle packing are also integral
- Expect the number of circles of curvature $\leq N$ grows like

$$CN^{\delta}, \quad \delta \approx 1.41$$

where $C$ depends on the packing, $\delta$ is absolute

# Modular obstructions

**Proposition**

The odd curvatures are all equivalent modulo 4.

**Definition**

An integer is *admissible* if it satisfies all modular restrictions in a given packing.

**Theorem**

*In a primitive Eisenstein circle packing, the set of curvatures of circles is density 1 in the set of admissible integers.*

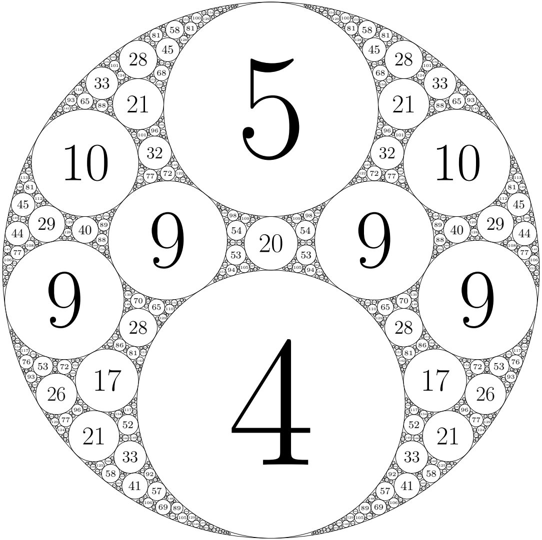
# Local-global

**Question**

Do all sufficiently large admissible integers appear as curvatures?

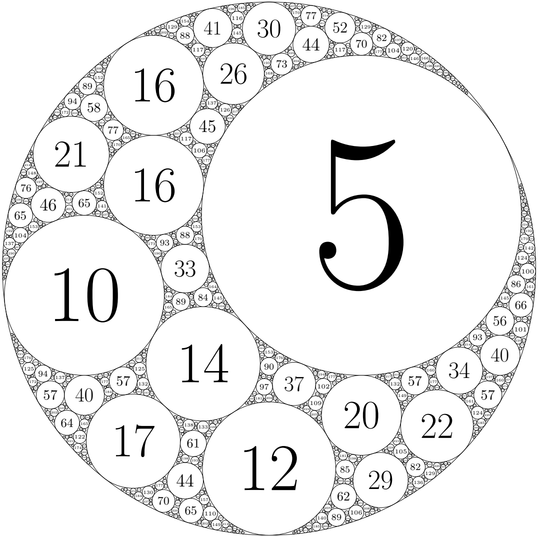
- This was conjectured for Apollonian circle packings
- Proven false in 2023: there can exist families like $n^2$ or $n^4$ which are are ruled out by quadratic/quartic reciprocity (Haag-Kertzer-R.-Stange)
- Rough outline for some cases:

  *The Kronecker symbol of coprime tangent circle curvatures is constant across the packing.*

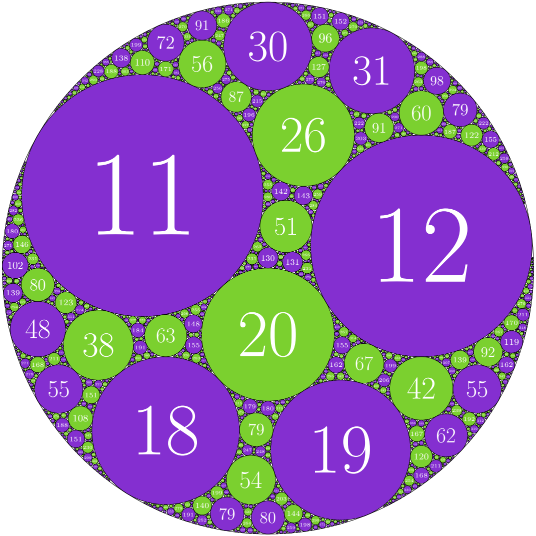



Expect local-global to be true

# Example 2




Missing families $(2n + 1)^2, 2n^2, 6n^2$




Green misses $2n^2, 3(2n+1)^2$; purple misses $6n^2$

# Moieties

### Definition

The full tangency graph is bipartite. A moiety is one of these halves.

### Theorem

*There is a well-defined invariant $\chi_2$ on a moiety of a packing. If the extended type of the moiety is not $(3,1,1)$, then the local-global conjecture is false on the moiety.*

### Remark

This breaking into moities also occurs in the work of Shi-Shi-Whitehead-(Williams-Tracy)-Zhang on octahedral, cubic, square, and triangular packings.

# ¿Cubic reciprocity?

- Our obstructions only use quadratic reciprocity in a similar way to the Apollonian case
- Extensive computation suggests that we have found all non-modular obstructions
- Tragically, this would mean there are no cubic obstructions
- Possible issue: lack of control over curvatures modulo 3

# Thank you for listening!

Paper: `https://arxiv.org/abs/2605.16053`

PARI/GP package:
`https://github.com/JamesRickards-Canada/Eisenstein`